Proposition: \( f_0 \in B^* \) = Banach space \( \| f_0 \| = M. \)

Then \( f \) is continuous linear functional \( f \) on \( B \) = \( \mathbb{R}^n \), \( f(f_0) = M \) and \( \| f \|_{B^*} = 1 \).

Proof: Let's begin with example. Take \( B = \mathbb{R}^n \),
\( f_0 = (v_1, v_2, \ldots, v_n) \). All the linear functionals are of the form \( x \rightarrow x \cdot \alpha \), \( \alpha \in \mathbb{R}^n \). We want \( v \cdot \alpha = |v| \) and \( \| f \|_{B^*} = 1 \).

Taking \( \alpha = \frac{v}{|v|} \) does the job.

Now take \( B = L^4((0, 1)) \), \( f_0(x) = x \) viewed as an element of \( L^4((1)) \).

It is enough to construct \( g \in L^{\frac{4}{3}}((1)) \) s.t.
\[
\int_0^1 x^4 g(x) \, dx = \left( \int_0^1 x^4 \, dx \right)^{\frac{4}{5}} = \left( \frac{1}{5} \right)^4
\]
and \( \int_1^0 g(x)^{\frac{4}{3}} \, dx = 1. \) It is pretty clear that we can play around w/ polynomials and make it happen, but this shows how non-trivial the calculation can be without a concrete plan.
We now move on to a proof. Define \( l_0 \) on the one-dimensional subspace \( \mathbb{R} \) by \( l_0(\alpha f_0) = \alpha M \), for each \( \alpha \in \mathbb{R} \).

Set \( p(f) = \| f \| \) for each \( f \in B \), sub-linearity is satisfied. Observe that

\[
|l_0(\alpha f_0)| = |\alpha| M = |\alpha| \| f_0 \| = p(\alpha f_0),
\]

so \( l_0(f) \leq p(f) \) on this sub-space.

By Hahn–Banach, \( l_0 \) extends to \( B \) with

\[
l(f) \leq p(f) = \| f \|.
\]

The inequality also holds for \( f_f \). So

\[
1 \leq \| f \| 
\rightarrow \| f \|_{B^*} \leq 1.
\]

Since \( l(f_0) = \| f_0 \|_0 \), \( \| l \|_{B^*} \geq 1 \).

An application to linear transformations:

\( B_1, B_2 \) Banach spaces, \( T: B_1 \rightarrow B_2 \)

bounded linear transformation

\[
\| T \|_{B_2} \leq M \| T \|_{B_1} \quad \forall f \in B_1,
\]

uniform \( \| T \| = \text{least } M \) above.
Proposition: \( B_1, B_2 \) Banach spaces, \( S \subseteq B_1 \) dense linear subspace

Suppose that \( T_0 : S \to B_2 \) linear \( w/ \)

\[ \| T_0 s \|_{B_2} \leq M \| s \|_{B_1} \quad \forall s \in S \]

Then the same bound holds for all \( f \in B_1 \) in the sense that there is a unique extension \( \overline{T} : B_1 \to B_2 \).

Proof: If \( f \in B_1 \), let \( \overline{T} f = S \to f \to f \).

Observe that \( \| T_0 (S) - \overline{T} (S_n) \|_{B_2} \leq M \| S_n - f_m \|_{B_1} \)

\( \overline{T} S \) is Cauchy, it converges to a limit, denoted by \( \overline{T} f \) and we are done.

\[ \overline{T} : B_2 \to B_1 \]

Duality: If \( T : B_1 \to B_2 \), it induces \( T^* : B_2 \to B_1 \) defined as follows.

Let \( \ell_2 \in B_2 \). Then \( \ell_1 = T^* (\ell_2) \in B_1^* \):

\[ \ell_1 (\ell_2) = \ell_2 (T \ell_1) \]

As it turns out, the norms of \( T \& T^* \) are the same. This is incredibly useful for applications.
Theorem: Suppose that \( T: B_1 \to B_2 \) is a bounded linear transformation. Then \( T^*: B_2^* \to B_1^* \) is a bounded linear transformation with norm \( \|T^*\| = \|T\| \).

Proof: If \( \|f\|_{B_1} \leq 1 \),

\[
\|f(Tg)\|_{B_2} = \|T^*f(g)\|_{B_2} \leq \|T^*\| \|f\|_{B_1} \|g\|_{B_2} \leq \|T^*\| \|g\|_{B_2}.
\]

In other words, \( T^*g \in B_1 \), so \( \|T^*g\|_{B_1} \leq \|T^*\| \|g\|_{B_2} \).

\[
\|T^*\| = \sup_{\|g\|_{B_2} = 1} \|T^*g\|_{B_1} \leq \|T^*\| \|g\|_{B_2} \text{ which implies that } \|T^*\| \leq \|T\|.
\]

The converse is much more interesting. Assume \( \|T^*\| \leq \|T\| \).

Given \( \varepsilon > 0 \) we can find \( g \in B_1 \) w/ \( \|g\|_{B_1} = 1 \) and \( \|T^*(g)\|_{B_2} \geq \|T\| - \varepsilon \).

Let \( f = T^*(g) \in B_2 \). Then by a proposition above,

\[
\exists \; \ell_2 \in B_2^* \; \text{ s.t. } \|\ell_2\|_{B_2^*} = 1 \; \text{ and } \; \ell_2(f) \geq \|T\| - \varepsilon.
\]

By definition of \( T^* \),

\[
T^*(g) = \ell_2(Tg) = \ell_2(f) \geq \|T\| - \varepsilon.
\]

Since \( \|g\|_{B_1} = 1 \), \( \|T^*\| \geq \|T\| - \varepsilon \Rightarrow \|T^*\| \geq \|T\| \).