We are ready to roll...

\[
\frac{f(a+h) - f(a)}{h} = \sum_{i=1}^{n} \left( f(a_i^+, a_{i+1}^-, ..., a^n) - f(a_i^-, a_{i+1}^+, ..., a^n) \right) + \sum_{i=1}^{n} \left( f(a_i^+, a_{i+1}^-, ..., a^n) - f(a_i^-, a_{i+1}^+, ..., a^n) \right) + \sum_{i=1}^{n} \left( f(a_i^+, a_{i+1}^-, ..., a^n) - f(a_i^-, a_{i+1}^+ + h^n, a^n) \right).
\]

Let \( g(x) = f(x, a^2, ..., a^n) \). By mean-value theorem,

\[
g(a+h, a^2, ..., a^n) - g(a, a^2, ..., a^n) = h \cdot D_1 g(b_1, a^2, ..., a^n)
\]

for some \( b_1 \in (a, a+h) \).

Proceeding in the same way, the \( i \)th term above equals

\[
h \cdot D_i g(a^+, a^2, ..., b_i^-, ..., a^n) = h \cdot D_i g(c_i)
\]

for some \( c_i \in (a, a+h) \).

It follows that

\[
\lim_{h \to 0} \frac{f(a+h) - f(a) - \sum_{i=1}^{n} D_i f(c_i) \cdot h^i}{h} = \lim_{h \to 0} \sum_{i=1}^{n} [D_i f(c_i) - D_i f(a)] \cdot h^i
\]

for \( i = 1 \).
$$\lim_{|h| \to 0} \frac{1}{|h|} \sum_{i=1}^{n} \left| D_i f(a_i) - D_i f(a) \right|$$

since $D_i f$ is continuous at $a$.

Here is a very useful application of the chain rule:

**Theorem:** Let $g_1, g_2, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ continuously and let $F : \mathbb{R}^m \to \mathbb{R}$ be differentiable at $(g_1(a), \ldots, g_m(a))$. Define $F : \mathbb{R}^m \to \mathbb{R}$, $F(x) = f(g_1(x), \ldots, g_m(x))$. Then

$$D_i F(a) = \sum_{j=1}^{m} D_i f(g_j(a), \ldots, g_m(a)) \cdot D_i g_j(a)$$

**proof:** $F = f \circ g$, $g = (g_1, g_2, \ldots, g_m)$ differentiable at $a$.

$$F'(a) = f'(g(a)) \circ g'(a) =$$

$$(D_i f(g(a)), \ldots, D_m f(g(a))) \cdot (D_i g_1(a), \ldots, D_m g_1(a))$$

and we are done! (why?)

$$(D_i g_m(a), \ldots, D_m g_m(a))$$