proof of the three-lines lemma:

Assume $M_0 = M_1 = 1$, $\sup_{0 \leq x \leq 1} |\Phi(x+iy)| \to 0$ as $|y| \to \infty$.

Let $M = \sup_{z \in \text{closure of the strip } S} |\Phi(z)|$, note that the rate of convergence is not specified, so this assumption is far more general than it seems.

Consider a sequence $(\varepsilon_n) \to M$ as $n \to \infty$.

The points cannot go to infinity by assumption. It follows that a subsequence of $(\varepsilon_n)$ converges to some $\varepsilon \in S$ by the maximum principle, so $\varepsilon$ is boundary of $S$, so $M \leq 1$ since $|\Phi| \leq 1$ on the boundary, so

Now assume that $M_0 = M_1 = 1$ and define

$$
\Phi_\varepsilon(z) = \Phi(z) e^{\varepsilon(z^2-1)}, \quad \varepsilon > 0.
$$

Observe that $(x+iy)^2 - 1 = x^2 - 1 - y^2 + 2xyi$.

It follows that $|\Phi_\varepsilon(z)| \leq 1$ on $\Re(z) = 0$, $\Re(z) = 1$, and $\sup_{0 \leq x \leq 1} |\Phi_\varepsilon(x+iy)| \to 0$ as $|y| \to \infty$ for each $\varepsilon > 0$.

Taking $\varepsilon \to 0$ gives $|\Phi| \leq 1$ as needed.

Now let's deal with the case if general $M_0$, $M$. Let $\Phi(z) = M_0^{x^2-1} M_1^y \Phi(z)$. All the conditions are
satisfied and we are done.

We are now ready to proceed to the proof of the interpolation theorem. Suppose that $f$ is simple and $\|f\|_p = 1$.

In order to prove that $\|f\|_q \leq M \|f\|_p$, it is enough to show that

$$
\int_I g \, du \leq M \|f\|_p \|g\|_q', \quad \frac{1}{p} + \frac{1}{q'} = 1
$$

simple, $\|g\|_q' = 1$

We first handle the case $p < \infty$ and $q > 1$. Recall that we are taking $\|f\|_p = 1$ and define

$$
g_2 = |S| \frac{f_2}{S} \quad \text{where} \quad S(\tau) = R \left( \frac{1 - \tau}{P_0 + \tau} \right)
$$

$$
\text{and} \quad g_2 = |g| \frac{S(\tau)}{|g|} \quad \text{where} \quad S(\tau) = R \left( \frac{1 - \tau}{P_0 + \tau} \right)
$$

Note that since $\frac{1}{p} = \frac{1 - \tau}{P_0 + \tau}$, $\frac{1}{q'} = \frac{1 - \tau}{P_0 + \tau}$

$$
f = g, \quad g_2 = g. \quad \text{Also,}
$$

$$
\left\{ \begin{array}{l}
\|g\|_p = 1 \quad \text{if} \quad \text{Re}(\tau) = 0 \\
\|g_2\|_p = 1 \quad \text{if} \quad \text{Re}(\tau) = 1
\end{array} \right.
$$
Similarly,

\[
\begin{cases}
\|g_e\|_{\mathcal{E}_0} = 1, & \text{Re}(\xi) = 0 \\
\|g_e\|_{\mathcal{E}_1} = 1, & \text{Re}(\xi) = 1
\end{cases}
\]

And here comes the main idea. Let

\[\Phi(\xi) = S(T^{\xi}) g_e d\nu.\]

Since \( g = \sum a_k \chi_{\mathcal{E}_k} \), \( f_2 \) is also simple:

\[ f_2 = \sum \frac{1}{\|a_k\|_{\mathcal{E}_1}} \frac{a_k}{S(\xi)} \chi_{\mathcal{E}_k} \]

Similarly, \( f_3 = \sum \frac{1}{\|b_k\|_{\mathcal{E}_1}} \frac{b_k}{S(\xi)} \chi_{\mathcal{E}_k} \).

It follows that

\[ \Phi(\xi) = \sum_{j,k} \frac{a_k}{\|a_k\|_{\mathcal{E}_1}} \frac{b_k}{\|b_k\|_{\mathcal{E}_1}} \frac{S(\xi)}{S(\xi) \chi_{\mathcal{E}_k} d\nu} \]

holomorphic on the strip and continuous on the boundary.

By Hölder, \( |\Phi(\xi)| \leq \|T^{\xi}\|_{\mathcal{E}_0} \|g_e\|_{\mathcal{E}_0} \leq M_0 \|T^{\xi}\|_{\mathcal{E}_0} = M_0, \) if \( \text{Re}(\xi) = 0. \)

Similarly, \( |\Phi(\xi)| \leq M_1, \) if \( \text{Re}(\xi) = 1. \)
By the three-lines lemma, we are done for simple functions. The general case proceeds by approximation. Please fill in the details.

We have not yet handled the case \( p = 1 \) and \( p = \infty \).

If \( p = \infty \), \( p_0 = p_1 = \infty \). We have

\[
\|Tg\|_{L^p} \leq M_0 \|g\|_{L^\infty} \quad \text{and} \quad \|Tg\|_{L^p} \leq M_1 \|g\|_{L^\infty}
\]

The rest follows by Hölder's inequality (check), i.e.

\[
\|Tg\|_{L^q} \leq \|Tg\|_{L^p} \cdot \|\sigma\|_{L^{p/q}}
\]

If \( \varepsilon = 1 \), then \( p_0 = p_1 = 1 \). So just take \( q_2 = q \). We run the same argument as above.

Applications!

Young: \( \|f \ast g\|_{L^p} \leq \|f\|_{L^{p/q}} \cdot \|g\|_{L^q} \), \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 \), \( \infty \geq q \geq 1 \)

Proof: Consider \( T(f) = f \ast g \). Then

\[
\int (f \ast g(x)) \, dx = \int \int f(x-y)g(y) \, dy \, dx
\]

\[
\leq \|f\|_{L^{p/q}} \cdot \|g\|_{L^q}, \quad \text{so} \quad T: L^p \rightarrow L^q \text{ is norm preserving.}
\]