Implicit Functions

Suppose \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is defined by \( f(x,y) = x^2 + y^2 - 1 \). If we choose \((a,b)\) with \( f(a,b) = 0\), \( a \neq 1 \) or \(-1\), there are open intervals \( A, B \) with \( a \in A, b \in B \) with the following property: if \( x \in A \), there is a unique \( y \in B \) such that \( f(x,y) = 0 \)

- The functions \( g \) and \( g' \) are defined implicitly by the function \( f(x,y) = 0 \)
- What happens if we choose \( A \) to contain \( 1 \) or \(-1\)?
Question: If \( f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \), and \( f(a_1, \ldots , a_n, b) = 0 \), when can we find, for each \((x_1, \ldots , x_n)\) near \((a_1, \ldots , a_n)\) a unique \( y \) near \( b \) such that \( f(x_1, \ldots , x_n, y) = 0 \)?

More generally: Suppose we have \( m \) equations depending on \( x_1, \ldots, x_n \), in \( m \) variables, \( \phi = f \colon \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) and \( i = 1, \ldots, m \), when can we find, for each \( x \) near \( a \), a unique \( y \) near \( b \) satisfying \( f_i(x, y) = 0 \)?

The Implicit Function Theorem:

- Suppose \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) is continuously differentiable in an open set containing \((a, b)\), and \( f(a, b) = 0 \). Let \( M \) be the \( m \times m \) matrix \((\partial_{x_i} f_i(a, b))\) where \( 1 \leq i \leq m \).

- If \( \det M \neq 0 \), there is an open set \( A \subset \mathbb{R}^n \) containing \( a \) with the following property: for each \( x \in A \) there is a unique \( y(x) \in \mathbb{B} \) such that \( f(x, y(x)) = 0 \).

- The function \( y(x) \) is differentiable.
Proof: (We create this $F$ just to get a function whose Jacobian is a square)

1. Define $F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$.

   Then $\det F' = \det M \neq 0$. (Here $F'$ is just the Jacobian of $F$.)

2. Next we apply the inverse function theorem: There is an open set $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a, b) = (a, 0)$ and an open set in $\mathbb{R}^n \times \mathbb{R}^m$ containing $(a, b)$ which we may take to be of the form $A \times B$, such that $F: A \times B \to W$ has a differentiable inverse $h: W \to A \times B$.

   What does $h$ look like?

   $h(x, y) = (x, k(x, y))$


\[ f(x, k(x, y)) = \]

\[ (\pi_0 \circ F) \circ h(x, y) = \]

\[ \pi_0 \circ F \circ h(x, y) = \]

\[ \pi_0 \circ F \circ h(x, y) = \]

\[ \pi(x, y) = y \]

\[ f(x, k(x, y)) = 0 \Rightarrow q_\delta(x) = k(x, y) \]

\[ \square \]
What's the derivative of $q_\delta$?

$f^i(x, q_\delta(x)) = 0$ by hypothesis

$\Rightarrow \quad 0 = D_q f^i(x, q_\delta(x)) + \sum_{a=1}^{n+1} D_{x^a} f^i(x, q_\delta(x)) \cdot D_i q^a(x) \quad i, j = 1, \ldots, m$

$\cdot \text{Det} \, M \neq 0 \Rightarrow \text{we can solve for } D_q q^a(x)$.

Example:

$\cdot f(x, y) = x^2 + y^2 - 1 \Rightarrow q(x) = \sqrt{1-x^2} + q_\delta(x) = -\sqrt{1-x^2}$ satisfy $f(x, q(x)) = 0$.

$\Rightarrow \quad 0 = D_1 f(x, q(x)) + D_2 f(x, q(x)) \cdot q'(x) = 0$

$= 2x + 2q(x) \cdot q'(x) = 0 \quad \Rightarrow \quad q'(x) = -\frac{x}{q(x)}$
A generalization

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable in an open set containing $a$ where $p \leq n$. If $f(a) = 0$ and the $p \times n$ matrix $(D_i f^j(a))$ has rank $p$, then there is an open set $A \subset \mathbb{R}^n$ containing $a$ and a differentiable function $h : A \to \mathbb{R}^n$ with differentiable inverse such that

$$f \circ h(x^1, \ldots, x^n) = (x^{n-p+1}, \ldots, x^n)$$

How can we see the previous theorem as a special case?

How can we modify the proof for this situation?

Regard $f$ as a function

$$f : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$$
Example:

Let \( f_1 : \mathbb{R}^3 \rightarrow \mathbb{R} \) be \( f_1(x, y, z) = x^2 + y^2 + z^2 - 1 \)
and \( f_2 : \mathbb{R}^3 \rightarrow \mathbb{R} \) be \( f_2(x, y, z) = x^2 + y^2 - y \).

Let \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be \( F(x, y, z) = (f_1(x, y, z), f_2(x, y, z)) \).

Then the set \( \{ a \in \mathbb{R}^3 : F(a) = 0^2 \} \) is the intersection of the two surfaces \( f_1 = 0 \) and \( f_2 = 0 \).

\[
\text{JF} = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}
\end{pmatrix}
= \begin{pmatrix}
2x & 2y & 2z \\
2x & 2y & 2z - 1 & 0
\end{pmatrix}
\]

At what points is \( \text{JF} \) less than or equal to rank 2?

What is this telling us geometrically?
Notation

This notation is referred to a "classical":

- $D_1 f(x,y,z) \quad \frac{df}{dx} \quad \frac{df}{dx} (x,y,z) \quad f_x$
- $D_2 f(x,y,z) \quad \frac{df}{dx} \quad \frac{df}{dx} (x,y,z) = \frac{df}{dx} (y,x,z)$
- $D_3 f(x,y,z) \quad \frac{df}{dx} \quad \frac{df}{dx} (x,y,z)$

For $f: \mathbb{R} \to \mathbb{R}$, everyone agrees that we use "$d" not "\partial": \frac{d \sin x}{dx}$

Want to be confused? Try stating the implicit function theorem using "classical" notation.
Chapter 3: Integration

- The basic definitions here are exactly the same as the ones for single-variable calculus, just with more numbers.

- For example, a partition of a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ works exactly how you'd expect: $I + \mathcal{P} = \{ P_i \}_{i=1}^{n} = \mathcal{P}$ is a list of partitions for $[a_i, b_i]$.

- Now suppose that $A$ is a rectangle and $f: A \to \mathbb{R}$ is a bounded function with partition $P$ of $A$. Let $S$ be a subrectangle. Then we have:

  $m_S(f) = \inf \{ f(x) : x \in S \}$
  $M_S(f) = \sup \{ f(x) : x \in S \}$
  $v(S) = \text{volume} = \prod_{i=1}^{n} (b_i - a_i)$

  $L(f, P) = \frac{1}{5} \sum_{i=1}^{n} m_S(f) \cdot v(S)$
  $U(f, P) = \frac{1}{5} \sum_{i=1}^{n} M_S(f) \cdot v(S)$
  $L(f, P) \leq U(f, P)$ for any two partitions $P, P'$. 
Definition: A function \( f: A \rightarrow \mathbb{R} \) is called integrable on the rectangle \( A \) if \( f \) is bounded and \( \sup \{ L(f, P) \} = \inf \{ U(f, P) \} \). This number is denoted by \( \int_A f \) or \( \int f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \) and is called the integral of \( f \) over \( A \).

Theorem: A bounded function \( f: A \rightarrow \mathbb{R} \) is integrable if and only if for every \( \varepsilon > 0 \) there is a partition \( P \) of \( A \) such that \( U(f, P) - L(f, P) < \varepsilon \).