Proposition: If $\psi \in \mathcal{S}$, then $F \ast \psi$ is a slowly increasing $C^\infty$ function and

$$ F \ast \psi = \psi \ast F. $$

Proof: By above, $|F(\psi(x))| \leq c \|\psi\|_N = O(1|x|^{-N})$. Now, $\|\psi^\ast \psi \|_N \leq c (1+1|x|)^N \|\psi\|_N + |x| \rightarrow \infty$ and we are done.

To get the formula, note that

$$(F \ast \psi)\hat{\phi} = F(\hat{\psi} \ast \phi) \iff (F \ast \psi)\hat{\phi} = F(\hat{\psi} \ast \phi)$$

Also, $\hat{\psi^\ast \psi} = F(\hat{\psi} \ast \phi) = F(\hat{\psi} \ast \phi)\hat{\phi}$

and we are done since $(\hat{\psi} \ast \phi)\hat{\phi} = \hat{\psi} \ast \phi$ (direct calculation).

Proposition: If $F$ is a distribution of compact support, then its Fourier transform $F^\wedge$ is a slowly varying $C^\infty$ function.

In fact, $F(\epsilon) = F(\xi)$, where $\xi = \eta(x) e^{2\pi i x \cdot \epsilon}$

in $\mathcal{D}'$ and $\epsilon = 1$ in the neighborhood of $\text{supp}(F)$

Proof: $|F(\xi)| \leq c \|\xi\|_N \leq c (1+1|x|)^N$.

Similarly, $|\partial_\xi^\alpha F(\xi)| \leq c_\alpha (1+1|x|)^N + |x|^\alpha$,

ie $F(\xi)$ is $C^\infty$ and slowly increasing.

It suffices to prove that $\int \mathcal{D}' \left[ F(\xi) \phi(\xi) d\xi \right] = F(\phi)$.

$\check{\text{PED}}$ extend by density.
Let $F = \delta^\varepsilon$ measure on $S'$.

$$F(\xi) = \int e^{-2\pi i x \cdot \xi} \eta(x) \, db(x) = \hat{\eta} \delta(\xi) = \hat{\delta}(\xi)$$

$$\int \hat{\delta}(\xi) \varphi(\xi) \, d\xi = \int e^{-2\pi i x \cdot \xi} \varphi(\xi) \, db(x) = \int \hat{\varphi}(x) \, db(x) = F(\hat{\varphi})$$

Let $g(\xi) = F(\xi) \varphi(\xi)$ is continuous, compact support.

$$\int_{\mathbb{R}^d} F(\xi) \varphi(\xi) \, d\xi = \int_{\mathbb{R}^d} g(\xi) \, d\xi = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} S_\varepsilon$$

$$S_\varepsilon = e^{i \varepsilon \sum_{n \in \mathbb{Z}^d} g(n \varepsilon)}$$

Observe that $S_\varepsilon = F(S_\varepsilon)$, w.l.o.g.

$$S_\varepsilon = e^{i \sum_{n \in \mathbb{Z}^d} \xi_n \eta(x) \varphi(n \varepsilon)} = \eta(x) \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} \varphi(\xi) \, d\xi$$

in $\|\cdot\|_W$ norm.

It follows that $S_\varepsilon \Rightarrow F(\hat{\eta} \varphi) = F(\hat{\varphi})$ since $\eta = 1$ in the neighborhood of the support of $F$.

This completes the proof.
\[
\frac{d}{dx} K(x) = \int_{-\infty}^{\infty} f(x) \, dx
\]

Theorem: Suppose \( F \) is a distribution supported at the origin.
simple function

If the limit exists at \( a \in \mathbb{R} \),

\[
\lim_{x \to a} f(x) = L
\]

at a point \( a \), then \( f \) is continuous at \( a \).

Cauchy criterion: \( f \) is continuous at \( x = a \) if

\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.
\]

We can now apply the Cauchy criterion to the function \( f \).

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

Since \( f \) is continuous at \( a \),

\[
f'(a) = \lim_{x \to a} f'(x)
\]

Choose \( q = \frac{1}{2} \).

By the definition of the limit, \( \lim_{x \to a} f(x) = L \).

When \( \varepsilon > 0 \), the limit of \( f(x) \) as \( x \to a \) is \( L \),

there exists \( \delta > 0 \) such that for all \( x \),

\[
|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.
\]

Conclusion of the proof: Apply the Cauchy criterion to \( f \).

We are done.
\[ \frac{d}{dx} \left( \log(x) \right) = \frac{\log(x) - \log(y)}{x - y} \]

Hence, the distribution \( p(x) \) equals:

\[ f(x) = \frac{\log(x) - \log(y)}{x - y} \]

\[ \frac{\log(x)}{x} \]

Indeed, \( \int_{0}^{\infty} f(x) \, dx = \int_{0}^{\infty} x^{-\frac{1}{2}} e^{-x} \, dx \]

\[ = \frac{\pi}{2} \]

\[ = \frac{\pi}{2} \int_{0}^{\infty} x^{-\frac{1}{2}} e^{-x} \, dx \]

\[ = \frac{\pi}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \]

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\[ = \frac{\pi}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \]
\( \mathcal{L} \) is homogeneous in \( \mathbb{R}^n \) and \( \mathbb{R}^m \)

\[
\mathcal{L}(\mathcal{F}(x)) = \frac{\partial}{\partial x} x_0 u(x) \frac{\partial}{\partial x} \mathcal{F}(x) = x_0 \mathcal{F}(x)
\]

Suppose that \( \mathcal{F}(x) = \mathcal{F}_0(x) \mathbb{1}_{x < a} \)

Homogeneous if \( \mathcal{F}_0(x) \mathbb{1}_{x < a} = \frac{x}{a} \mathcal{F}_0(x) \mathbb{1}_{x < a} \)

If \( \mathcal{F} \) is a distribution, \( \mathcal{F}(x) = \mathcal{F}_0(x) \mathbb{1}_{x < a} \)

\( \mathcal{L} \) is a drift term

\( \mathcal{F}_0(x) \mathbb{1}_{x < a} \) is a function of \( a \)

Homogeneous distributions:
It is clear that $\xi$ is homogeneous of degree 1, e.g., what

This suggests that the Fourier transform of the distribution

$$f(x, \lambda) = \hat{p}(\lambda) f_0,$$

where $p(x) = \mathbb{1}_{[-1, 1]}(x)$.

Suppose that $f(x, \lambda) = \mathbb{1}_{[-1, 1]}(x) f_0$. Then, the Fourier transform is

$$f(x, \lambda) = \hat{p}(\lambda) f_0.$$
Let \( e \in \mathbb{R} \) be a point.

Also, \( \sqrt{a} = \sqrt{a} \) as we saw earlier.

From (3), \( e = \sqrt{a} \)

Proof: \( \sqrt{a} = \sqrt{a} \)

Notice that

\[
\frac{\frac{d}{dx}}{\sqrt{a}} = \frac{d}{dx} \left( \sqrt{a} \right)
\]

Theorem: \( d > 0 \), then

Let \( H \) denote the distribution corresponding to \( \sqrt{a} \).

Therefore, \( \sqrt{a} = \sqrt{a} \)

(3)

Therefore, \( \sqrt{a} = \sqrt{a} \)

By definition, \( \sqrt{a} = \sqrt{a} \)

Since \( \sqrt{a} \), the Fourier transform \( F \) is homogeneous.

Problem: Find the distribution in IR that is homogeneous.
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \pi \]

**Lemma:** If \( f(x) \) is continuous and \( f(x) \to 0 \) as \( |x| \to \infty \), then

\[ \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = 0 \]

**Proof:** Let \( R > 0 \) be arbitrary. Then

\[ \int_{-R}^{R} f(x) \, dx = \int_{-R}^{0} f(x) \, dx + \int_{0}^{R} f(x) \, dx \]

Since \( f(x) \to 0 \) as \( |x| \to \infty \), we have

\[ \int_{-R}^{0} f(x) \, dx \to 0 \quad \text{and} \quad \int_{0}^{R} f(x) \, dx \to 0 \]

as \( R \to \infty \). Therefore,

\[ \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx = 0 \]

as desired.