Theorem: \( x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}^n, a \in \mathbb{R} \), then

i) \( \langle x, y \rangle = \langle y, x \rangle \)

ii) \( \langle ax, y \rangle = a \langle x, y \rangle, \langle x + x_2, y \rangle = \langle x, y \rangle + \langle x_2, y \rangle \)

iii) \( \langle x, x \rangle \geq 0 \) and \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \).

iv) \( |x| = \sqrt{\langle x, x \rangle} \)

v) \( \langle x, y \rangle = \frac{|x + y|^2 - |x - y|^2}{4} \)

**Proof:**

i) \( \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \langle y, x \rangle \)

ii) \( \langle ax, y \rangle = \sum_{i=1}^{n} ax_i y_i = a \langle x, y \rangle \)

\( \langle x + x_2, y \rangle = \sum_{i=1}^{n} (x_i + x_2) y_i = \langle x, y \rangle + \langle x_2, y \rangle \)

iii) \( \langle x, x \rangle = \sum_{i=1}^{n} x_i^2 \geq 0 \) if and only if \( x = 0 \)

iv) \( \sqrt{\langle x, x \rangle} = \sqrt{\left( \sum_{i=1}^{n} x_i^2 \right)^2} = |x| \)

v) \( \frac{|x + y|^2}{2} = \langle x + y, x + y \rangle = |x|^2 + |y|^2 + 2 \langle x, y \rangle \)

\( \frac{|x - y|^2}{2} = \langle x - y, x - y \rangle = |x|^2 + |y|^2 - 2 \langle x, y \rangle \)

The result follows.
Notation: \((0, \ldots, 0)\) denoted by \(0\)

\(e_1, \ldots, e_n\) standard bases for \(\mathbb{R}^n\)

\(T: \mathbb{R}^n \rightarrow \mathbb{R}^m\) linear transformation

with matrix \(A = \sum a_{ij} e_j\), \(m \times n\) matrix

\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{m1} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix}
= 
\begin{pmatrix}
  y_1 \\
  \vdots \\
  y_m
\end{pmatrix}
\]

\(T(e_i) = \sum_{j=1}^{m} a_{ij} e_j\)

We now study subsets of \(\mathbb{R}^n\):

\(A \subset \mathbb{R}^n, B \subset \mathbb{R}^m\)

\(A \times B = \{(x, y) : x \in A, y \in B\}\)

\((A \times B) \times C = A \times (B \times C)\)

Closed rectangle : \([a_1, b_1] \times \cdots \times [a_n, b_n]\)

Open rectangle : \((a_1, b_1) \times \cdots \times (a_n, b_n)\)

\(C \subset \mathbb{R}^n\) open if for each \(x \in C\), there is an open rectangle containing \(x\) that is contained in \(C\). \(C\) is closed if \(\mathbb{R}^n \setminus C\) is open.
If $A \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, one of the following must hold:

i) $\exists$ open rectangle $B \ni x \in B \cap A$

ii) """"""" $\exists x \in B \cap \mathbb{R}^n - A$

iii) If $B$ is any open rectangle w/ $x \in B$, then $B$
contains points of both $A$ and $\mathbb{R}^n - A$

i) $\Rightarrow$ interior of $A$ (open)

ii) $\Rightarrow$ exterior of $A$ (open)

iii) $\Rightarrow$ boundary of $A$ (closed; since the union of
two open sets is open)

A collection of open sets is an open cover of $A$ if every point $x \in A$ is in some member of this collection.

Example: $\exists (a, a+1)^2 : a \in \mathbb{R}$ covers $\mathbb{R}$, but no
finite sub-collection will do the job. The same is
ture for $\exists \left( \frac{1}{n}, 1 - \frac{1}{n} \right)^2$ with respect to $[0, 1]$.

A is compact if every open cover has a finite
subcover. We can characterize compactness as follows.
Theorem (Heine-Borel) The closed interval \([a, b] \subset \mathbb{R}\) is compact.

Proof: If \(O\) is an open cover of \([a, b]\), let \(A = \{x : a \leq x \leq b\}\) and \([a, x]\) is covered by some finite number of open sets in \(O\).

This set is clearly bounded by \(b\).

Claim: \(b \in A\). This is accomplished by establishing that \(\alpha = \text{ lub}(A) \in A\) and \(\forall \epsilon > 0\), \(\alpha - \epsilon < b\).

Since \(O\) is a cover, \(\alpha \in U\) for some \(U \in O\). Then all the points to the left of \(\alpha\) are also in \(U\).

Since \(\alpha = \text{lub}(A)\), \(\exists \epsilon > 0\), this interval \(\not\subset X \in A\). Thus \([a, x]\) is covered by a finite sub-collection of \(O\), while \([x, \alpha]\) is covered by \(U\). \([a, \alpha]\) is covered by a finite sub-collection.

To prove that \(b = \alpha\), suppose that \(\alpha < b\).

Then \(\exists x\) between \(\alpha\) and \(b\) \(\in [\alpha, x] \subset U\).

Since \(\alpha \in A\), \([\alpha, \alpha] = \emptyset\) is covered by finitely many elements of \(O\) \(\not\subset [\alpha, x'] \subset U\), contradiction!