Lemma: A closed rectangle \( f: A \to \mathbb{R} \) bounded \( \forall x \in A \), \( 0(f(x)) \leq c \).

Proof: For each \( x \in A \), \( \exists U_x \), closed rectangle \( U_x \),

\[ m(U_x) \leq m(U_x) < \varepsilon. \]

Extract a finite subcover (by \( U_x \)), say \( U_{x_1}, \ldots, U_{x_n} \) that cover \( A \).

Construct \( \mathcal{A} \) partition \( \mathcal{A} \) each \( S \in \mathcal{A} \), \( S \in \mathcal{A} \), for some \( i \).

Then \( M_S(s) - m_S(s) = \varepsilon \), \( \forall S \in \mathcal{A} \), so

\[ \begin{align*}
U(S, P) - L(S, P) &\geq \sum_{S \in \mathcal{A}} [M_S(s) - m_S(s)] \mu(A) \\
&< \varepsilon \mu(A)
\end{align*} \]

Food for thought: Can you construct a function and a family of rectangles \( \mathcal{A} \) illustrating that \( \mu(A) \) is the

Theorem (Serious Business) Suppose that \( B \) is \( \mathcal{E} \): \( f \) is not continuous at \( x \). Then

\[ f: A \to \mathbb{R} \]

\[ \text{closed rectangle} \]

Then \( f \) is integrable iff \( B \) is a set of measure 0.

Proof: The key idea is to "approximate" \( B \) using sets

\[ B_\varepsilon = \{ x \in A : \delta(x) > \varepsilon \} \]. Note that \( B_\varepsilon \subseteq B \implies B \) has measure 0.

Since \( B_\varepsilon \) is compact (proved in Chapter 1) \( B_\varepsilon \) also has content 0.

It follows that \( \exists \) finite collection \( U_1, U_2, \ldots, U_n \) of closed rectangles

whose interiors cover \( B_\varepsilon \) \( \varepsilon \sum_{i=1}^{n} \mu(U_i) \leq \varepsilon. \)
Construct a partition of $A$ as every $S \in \mathcal{P}$ is in one of two categories:

i) $\mathcal{S}_1 = \{ S : S \subseteq C_i \text{ for some } i \}$

ii) $\mathcal{S}_2 = \{ S : S \cap B_\varepsilon = \emptyset \}$

Let $|f(x)| < M$ for $x \in A$. Then $M_S(g) - m_S(g) < 2M$ for every $S$.

$$\sum_{S \in \mathcal{S}_1} \left( M_S(g) - m_S(g) \right) v(S) < 2M \sum_{i=1}^{n} v(U_i) < 2Me$$

If $S \in \mathcal{S}_2$, then $d(S, x) < \varepsilon$ by above.

$$\sum_{S' \in S} (M_{S'}(g) - m_{S'}(g)) v(S') < \varepsilon v(S) + \varepsilon \sum_{S \in \mathcal{S}_2} v(S')$$

It follows that

$$U(S, P') - L(S, P') = \sum_{S \in \mathcal{S}_1} \left( M_S(g) - m_S(g) \right) v(S')$$

$$+ \sum_{S' \in \mathcal{S}_2} \left( M_{S'}(g) - m_{S'}(g) \right) v(S')$$

$$< 2Me + \varepsilon v(A) \rightarrow \text{integrability}$$
Conversely, suppose that \( f \) is integrable. Since 
\[ B = B_1 \cup B_2 \cup \ldots \] 
it suffices to prove that each \( B_n \) has measure 0. This is because the union of
sets of measure 0 (countable union) has measure 0. The argument is straightforward.

Let \( \varepsilon > 0 \) be given. Let \( P \) = partition of \( A \),
\[ u(g, P) - l(g, P) < \frac{\varepsilon}{n}. \]
Let \( S \in \mathcal{S} \) rectangle \( P \)
(properly intersect \( B_n \))

If \( s \in S \), \( m_S(g) - m_S(g) \geq \frac{1}{n} \)
by definition of \( B_n \)

\[ \frac{1}{n} \sum_{s \in S} v(s) \leq \sum_{s \in S} \left[ m_S(g) - m_S(g) \right] v(s) \leq \sum_{s \in S} \left[ m_S(g) - m_S(g) \right] v(s) \leq \frac{\varepsilon}{n} \]

\[ \sum_{s \in S} v(s) \leq \varepsilon \]

\[ S \]

\[ B_n \text{ has measure 0}. \]