Baire Category Theorem and friends (proved in 1899 in René-Louis Baire's thesis)

$(X, d)$ metric space w/ distance $d$, topology induced by $d$ in the usual way.

Define $B_r(x) = \{ y \in X : d(x, y) < r \}$

$E^0$, $E \subseteq X$, the union of all open sets contained in $E$.

$\delta E$, $E \subseteq X$, the union of all open sets containing $E$.

$\overline{E}$, $E \subseteq X$, the intersection of all closed sets containing $E$.

If $E \subseteq X$, we say that $E$ is dense in $X$ if $\overline{E} = X$.

We say that $E$ is nowhere dense if $(\overline{E})^0 = \emptyset$.

Examples: A single point and Cantor set are both nowhere dense in $\mathbb{R}$.

The rationals are dense since $\overline{\mathbb{Q}} = \mathbb{R}$.
Categories: or meager

A set $E \subseteq X$ is of the first category if $E$ is a countable union of nowhere dense sets in $X$.

A set $E \subseteq X$ that is not of the first category is referred to as being of second category in $X$.

A set $E \subseteq X$ is called generic if its complement is of the first category.

Theorem: Every complete metric space $X$ is of the second category in itself.

Corollary: In a complete metric space, a generic set is dense.

Proof: Assume that $X = \bigcup_{n=1}^{\infty} F_n$, and each $F_n$ is closed by replacing each $F_n$ by its closure.

To obtain a contradiction, it suffices to find $x \in X \cap \bigcup_{n=1}^{\infty} F_n$. Since $F_1$ is closed and nowhere dense, $F_1$ is an open ball $B_{r_1}$ of radius $r_1$.

Since $F_2$ is closed and nowhere dense, $B_{r_1}$ cannot be completely inside $F_2$ since $F_2$ has an empty interior.
The process continues naturally, and we obtain a sequence of balls \( E_n \) with the following properties:

i) The radius of \( E_n \) tends to 0 as \( n \to \infty \).

The point here is that we can find a nested sequence \( E_1 \supset E_2 \supset \cdots \) of balls with radii \( r_n \) such that \( r_n < \frac{1}{2} r_{n+1} \) and \( E_1 \cap E_2 \neq \emptyset \).

ii) \( E_{n+1} \subset E_n \).

iii) \( F_0 \cap E_n \) is empty.

Choose any point \( x \in E_n \). Then \( x \notin \bigcup_{i=n}^{\infty} F_i \) and \( \bigcup_{i=n}^{\infty} F_i \) converges to a limit \( x \) since \( X \) is complete.

Since \( x \notin E_n \), \( x \notin F_n \) for all \( n \) and we have a contradiction.

The corollary is not entirely straightforward. Assume that \( E \) is generic but not dense. Then \( E \subset \bigcup_{i=n}^{\infty} B_i \) is a closed ball entirely contained in \( E \).

By genericity, \( E = \bigcup_{n=1}^{\infty} F_n \), \( F_n \) nowhere dense.

Since \( \bigcup_{n=1}^{\infty} \overline{F_n \cap B} \) is nowhere dense, each \( F_n \cap B \) is nowhere dense.
Theorem: Suppose that \( \{f_n\} \) is a sequence of continuous complex-valued functions on a complete metric space \( X \), and
\[
\lim_{n \to \infty} f_n(x) = f(x)
\]
each exists for every \( x \).

Then the set of points where \( f \) is continuous is a generic set.