Theorem: Let $\{f_n\}$ be a sequence of functions on a complete metric space $X$. Suppose that $f_n$ are continuous and complex valued. Assume that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$.

Then $f$ is continuous on a generic set.

Proof: Define $\text{osc}(f)(x) = \lim_{x \to 0} \omega(f)(x, x)$

$$\omega(f)(x, x) = \sup_{y, z \in B_r(x)} |f(y) - f(z)|$$

decreases w.r.t. so the limit exists.

Note that $\text{osc}(f)(x) < \varepsilon$ if $\exists B$ centered at $x$ so that $|f(y) - f(z)| < \varepsilon$ whenever $y, z \in B$.

Key facts:

i) $\text{osc}(f)(x) = 0$ if $f$ is continuous at $x$.

ii) $E_\varepsilon = \{ x \in X : \text{osc}(f)(x) < \varepsilon \}$ is open.

If $x \in E_\varepsilon$, then for $\varepsilon > 0$ and $y \in B_r(x)$, $\sup_{y \in B_r(x)} |f(y) - f(z)| < \varepsilon$.

It follows that if $x^* \in B_{\varepsilon/2}(x)$, then $x^* \in E_\varepsilon$ since

$$\sup_{y \in B_{\varepsilon/2}(x^*)} |f(y) - f(z)| \leq \sup_{y, z \in B_r(x)} |f(y) - f(z)| < \varepsilon.$$
Lemma: Suppose that \( \{f_n\} \) is a sequence of continuous functions on a complete metric space \( X \), and \( f_n(x) \to f(x) \) for each \( x \) as \( n \to \infty \).

Then, given an open ball \( B \subset X \) and \( \epsilon > 0 \), there is an open ball \( B_0 \subset B \) and an integer \( m \geq 1 \) so that
\[ |f_m(x) - f(x)| \leq \epsilon \quad \forall x \in B_0. \]

Proof: Let \( Y \subset B \)
- a closed ball; also a complete metric space

Define \( E_\epsilon = \{ x \in Y : \sup_{j, k \geq \ell} |f_j(x) - f_k(x)| \leq \epsilon \} \)

for a long in the sequence

Since \( f_n(x) \to f(x) \) for every \( x \),
\[ Y = \bigcup_{\epsilon = 1}^{\infty} E_\epsilon \]
Closed sets since \( \{ x \in Y : |f_j(x) - f_k(x)| \leq \epsilon \} \)
is closed by continuity of \( f_j, f_k \)

Now apply Baire category to \( Y \) to see that
some \( E_m \) above must contain an open ball \( B_0 \).

By design, \( \sup_{j, k \geq m} |f_j(x) - f_k(x)| \leq \epsilon \) whenever \( x \in B_0 \).
Now let $k \rightarrow \infty$, $j = m$ and the lemma is recovered. We are ready to sprint to the finish line.

Let $F_n = \{ x \in X : \text{osc}(\delta)(x) \geq \frac{1}{n^2} \}

\text{E}_c \cup \text{E}_n = \frac{1}{n^\infty}

Then $D = \bigcup F_n$

Discontinuities of $\delta$ we must show that each $F_n$ is nowhere dense.

Fix $n \geq 1$. It is enough to show that each $F_n$ has an empty interior. For the sake of contradiction, assume that $B$ is an open ball contained in $F_n$.

Set $\epsilon = \frac{1}{4n}$ and find $B_0 \subset B$ and $m \geq 1$ so that $|f_m(x) - f(x)| \leq \frac{1}{4n} \quad \forall x \in B_0$

Since $f_m$ is continuous, $\exists B' \subset B_0$.

$B' \setminus B$ $|f_m(y) - f_m(z)| \leq \frac{1}{4n} \quad \forall y, z \in B'$

$C$ $|f(y) - f(z)| \leq |f(y) - f_m(y)| + |f_m(y) - f_m(z)| + |f_m(z) - f(z)| = \overline{111}$
\[
\frac{1}{n}, \frac{1}{n} \leq \frac{1}{4n}, \quad \text{and} \quad \frac{1}{n} \leq \frac{1}{4n} \quad \text{since}
\]

by lemma

\[
|s_{m}(y) - s_{m}(z)| < \frac{1}{4n}
\]

by the continuity argument above.

\[
\Rightarrow |s(y) - s(z)| < \frac{3}{4n} < \frac{1}{n}
\]

\[
\Rightarrow \text{osc}(s)(x') < \frac{1}{n} \quad \text{if} \quad x' \text{ is the center of } \mathcal{B}.
\]

This contradicts the assumption that \( x' \in F_{0} \).

The following result is proved using analogous tricks:

**Theorem:** The set of functions in \( C([0,1]) \) that are nowhere differentiable is generic.

Please study this proof on your own as there may be a question about it on the take-home final.
The Uniform Boundedness Principle:

Suppose that $B$ is a Banach space and $\mathcal{L}$ is a collection of continuous linear functionals on $B$.

i) If $\sup_{\lambda \in \mathcal{L}} |\ell(\lambda)| < \infty$ for each $f \in B$, then

$$\sup_{\lambda \in \mathcal{L}} \|\ell\| < \infty.$$  

ii) This conclusion also holds if we only assume that $\sup_{\lambda \in \mathcal{L}} |\ell(\lambda)| < \infty$ for all $f$ in some set of second category.

Proof: It is enough to prove ii) by Baire category since $B$ is of second category.

For each $M$, define

$$E_M = \left\{ f \in B : \sup_{\lambda \in \mathcal{L}} |\ell(\lambda)| \leq M \right\}.$$ 

By assumption, $E = \bigcup_{M=1}^{\infty} E_M$ is a set of second category where

$$\sup_{\lambda \in \mathcal{L}} \|\ell\| < \infty.$$
Observe that each $E_m$ is closed, since

$$E_m = \bigcap_{\varepsilon \in \mathbb{E}} E_{m, \varepsilon}, \quad E_{m, \varepsilon} = \{ \xi : |\xi(z)| \leq M^2 \}$$

closed by continuity (of $\varepsilon$)

By Baire, some $E_{m_0}$ must have non-empty interior. In other words, there exists $\xi_0 \in \mathbb{E}$ and $r > 0$ such that $B_r(\xi_0) \subset E_{m_0}$. 

$$\Rightarrow \quad \forall \xi \in \mathbb{E}, \quad |\xi(z)| \leq M_0$$

whenever $\|\xi - \xi_0\| < r$.

$$\Rightarrow \quad \text{for } \|\xi\| \leq r \quad \text{and} \quad \forall \xi \in \mathbb{E},$$

$$\|\xi (g)\| \leq \|\xi (g + \xi_0)\| + \|\xi (-\xi_0)\| \leq 2M_0.$$ 

This concludes the proof.

We shall now apply this technology to the question of divergence of Fourier series.
\(B = C([-\pi, \pi])\) Banach space of continuous complex valued functions on \([-\pi, \pi]\) with sup norm \(\|f\| = \sup_{x \in [-\pi, \pi]} |f(x)|\).

Define \(a_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx\),

Fourier coefficients \(a_n \in \mathbb{C}\).

Fourier series of \(f\) : \(\sum_{n=-\infty}^{\infty} a_n e^{inx}\)

\(S_N(f)(x) = \sum_{n=-N}^{N} a_n e^{inx}\)

\(N^{th}\) partial sum

It is not difficult to see that

\(S_N(f)(x) = (f \ast D_N)(x)\), where

\(D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin[(N+1/2)x]}{\sin(x/2)}\)

Dirichlet Kernel

\(f \ast g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y) dy\)