Bonnoulli trials:

Players A and B flip a fair coin $N$ times. Each time $H$ (heads) appears, A wins a dollar. Each time $T$ (tails) appears, B wins a dollar.

There are $2^N$ possible sequences of outcomes. What are the chances of $A$ winning $k$ dollars, for some $k$?

We can answer this question by direct counting. Instead, we shall create a formalism that will allow us to do much more.

The $2^N$ outcomes can be viewed as points in $\mathbb{Z}_2^N$, the underlying probability space.

Indeed,

$$\mathbb{Z}_2^N = \{ x = (x_1, x_2, \ldots, x_N), x_j = 0 \text{ or } 1, 1 \leq j \leq N \}$$

$m = "probability" \ measure \ that \ assigns \ 2^{-N}\ to \ each \ point \ of \ \mathbb{Z}_2^N. \ Thus \ m(\mathbb{Z}_2^N) = 1.$

Let $E_n = \{ x \in \mathbb{Z}_2^N : x_n = 0 \}$. Then $m(E_0) = \frac{1}{2}, \quad 1 \leq n \leq N.$
Consider \( m(E_n \cap E_m) \): \( n \neq m \)

\[
E_n \cap E_m = \left\{ x \in \mathbb{Z}_2^N : x_n = x_m = 0 \right\} = \frac{1}{4} m(E_n) m(E_m)
\]

"independence"

We now introduce functions on our probability space. Probabilists call them "random variables."

Let \( \rho_n = \) amount player A wins (or loses) at the \( n \)th flip, i.e.

\[
\rho_n(x) = \begin{cases} 
1, & x_n = 0 \\
-1, & x_n = 1 
\end{cases}
\]

\[
S_N(x) = S(x) = \sum_{n=1}^{N} \rho_n(x)
\]

Total winnings of player A after \( N \) flips.

Let's now try to understand the probability that \( S(x) = k \), i.e. \( N_1 \) wins, \( N_2 \) losses and

\[
k = N_1 - N_2, \quad N = N_1 + N_2
\]

Note that \( k \) & \( N \) have the same parity.
Let $N$ be even (you will get an exercise on the odd case).

The number of $x$'s for which $S(x) = k$ is

\[
\binom{N}{N_i} = \frac{N!}{N_i!(N-N_i)!} = \frac{N!}{\binom{N+k}{2} \cdot \binom{N-k}{2}}
\]

It follows that

\[
m\left(\exists x: S(x) = k^2\right) = 2^{-N} \frac{N!}{\binom{N+k}{2} \cdot \binom{N-k}{2}}
\]

It is not difficult to see that maximum is achieved at $k = 0$, i.e.

\[
2^{-N} \frac{N!}{(\binom{N}{2})^2} \sim \frac{2}{\sqrt{2\pi}} N^{-\frac{1}{2}} \text{ by Stirling}
\]

\[
\implies 2^{-N}
\]

(This can be derived from

\[
\log n! = n \log n - n + O(\log n)
\]

We are ready to take the liberating limit as $N \to \infty$.)
\[ N = \infty \] We must dispense w/ some formalities first.

\[ X = \{ x = (x_1, x_2, \ldots, x_N, \ldots), \text{ each } x_n = 0 \text{ or } 1 \text{ for all } n \geq 1 \} \]

Definition: A set \( E \) is a cylinder set in \( X \) whenever there is a finite \( N \) \& \( E \subset \mathbb{Z}_2^N \), \( x \in E \) iff \((x_1, x_2, \ldots, x_N) \in E\).

The collection of all cylinder sets together w/ their finite unions and intersections forms an algebra, on \( X \) and complements.

Consequently, \( m(E) = m_N(E') \) extends to a measure on the \( \sigma \)-algebra of sets generated by the cylinder sets.

Notation:

\[ X = \text{ probability space} \quad m = \text{ probability measure} \]

Now let's go back to the functions \( r_n(x) \):

\[ r_n(x) = 1 - 2x_n \quad (-1 \text{ or } 1) \]

\[ x_n = 0 \text{ or } 1 \]
These functions set up a correspondence between $X$ and $[0, 1]$:

$$D : (x_1, \ldots, x_n, \ldots) \rightarrow \sum_{j=1}^{\infty} \frac{x_j}{2^j} = \xi \in [0, 1]$$

"digits"

nearly a bijection (why nearly?)

Claim: $E$ is the cylinder set:

$$E = \left\{ x : x_j = a_j, 1 \leq j \leq N \right\}$$

$0$ or $1$

Then $m(E) = 2^{-N}$, and

$D$ maps $E$ to $\left[ \frac{\ell}{2N}, \frac{\ell+1}{2N} \right]$,

$$\ell = \sum_{j=1}^{N} 2^{N-j} a_j$$

note that this interval has measure $2^{-N}$

This allows us to extend $r_0(x)$ to $r_0(f)$ undefined on a finite set.
Definition: $\exists f_0^3$ are mutually independent measurable functions on $X$.

$$\bigcap_{n=1}^{\infty} m(\{x: f_0(x) \in B_n\}) = \prod_{n=1}^{\infty} m(\{x: f_0(x) \in B_n\})$$

for any sequence of Borel sets.

We say that a collection of sets $\{E_n\}$ is mutually independent if their indicator functions are independent.

Cautionary note: Pair-wise independence does not imply independence of the whole collection. Please reproduce or look up a beautiful example due to Sergei Bernstein.
If \( I_n \)'s are bounded, mutual independence implies that

\[
\int_X f_1(x) \ldots f_n(x) \, dm = \frac{1}{n} \sum_{j=1}^n \int_X f_j(x) \, dm
\]

proved by taking \( f_j(x) = \chi_{E_j}(x) \) and then taking limits.

Example: \( (X, \mathcal{E}, m) \) is a product of \( (X_0, \mathcal{E}_0, m_0) \)'s, \( n=1,2,\ldots \), and \( m = \prod m_0 \)'s.

Suppose that \( f_n(x) = F_n(x_n) \), where each \( F_0 \) is given on \( X_0 \); \( X = (X_1, \ldots, X_n, \ldots) \).

Then \( \{E_n\} \)'s are mutually independent.

Indeed, let \( E_n = \{x : f_n(x) \in B_n\} \) and

\[
E'_n = \{x : F_n(x) \in B_n', E_n \subset X_n\}
\]

Then \( E_n = \{x : x_n \in E_n' \} \) is a cylinder set

with \( m(E_n) = m_0(E_n') \)

\[
\Rightarrow m\left( \bigcap_{n=1}^\infty E_n \right) = \prod_{n=1}^\infty m_0(E_n') = \prod_{n=1}^\infty m(E_n)
\]
Now take a limit as $N \to \infty$.

In particular, this shows that Rademacher functions are independent.

Our next goal is to understand the behavior of
\[
S_N(x) = \sum_{n=1}^{N} r_n(x)
\]
as $N \to \infty$.

This will soon lead us to the Central Limit Theorem for coin flips, due to De Moivre.