Theorem: If $B$ is compact and $O$ is an open cover of $\mathbb{R}^2 \times \mathbb{R}^2$, then there is an open set $\mathcal{U} \subset \mathbb{R}^2$ containing $x$ such that $\mathcal{U} \times B$ is covered by a finite number of sets in $O$.

We will prove this in a moment, but what is it for? Let $A, B$ be compact subsets of $\mathbb{R}^2, \mathbb{R}^m$, respectively. If $O$ covers $A \times B$, $O$ covers $\mathbb{R}^2 \times \mathbb{R}^2$ for each $x \in A$.

It follows by above that $\exists \mathcal{U}_x$ open, $x \in \mathcal{U}_x \ni \mathcal{U}_x \times B$ is covered by finitely many sets in $O$. Since $A$ is compact, a finite number $\mathcal{U}_{x_1}, \mathcal{U}_{x_2}, \ldots, \mathcal{U}_{x_n}$ cover $A$. Since finitely many sets of $O$ cover each $\mathcal{U}_{x_i} \times B$, finitely many sets in $O$ cover $A \times B$.

We conclude that if $A, B$ are compact, then $A \times B$ is compact.

Proof of Theorem: For each $y \in B$, $(x, y) \in W$ open in $O$. Therefore, $(x, y) \in \mathcal{U}_y \times V_y \subset W$ open rectangle.

The sets $\{V_y\}$ cover $B$, so by compactness, some finite number $V_{y_1}, \ldots, V_{y_n}$ cover $B$. Let $W = V_{y_1} \cap \ldots \cap V_{y_n}$. Then if $(x', y') \in A \times B$, we have $y' \in V_{y_i}$ for some $i$. 

Also, \( x' \in U_y \). Hence \((x', y') \in U_y \times V_y \subset W \) open \( \in O \).

Corollary: \( A_1, \ldots, A_k \) compact \( \Rightarrow \) \( A_1 \times A_2 \times \ldots \times A_k \) is compact.

Corollary: A closed bounded subset of \( \mathbb{R}^n \) is compact.

The converse is also true and is easier (Homework).

Proof: If \( A \subset \mathbb{R}^n \) is closed and bounded, then \( A \subset B \) for some closed rectangle \( B \). If \( O \) is an open cover of \( A \), then \( O \) is an open cover of \( B \) together with \( \mathbb{R}^n - A \).

Hence a finite number of sets \( U_1, \ldots, U_n \) in \( O \) together with \( \mathbb{R}^n - A \) cover \( B \). It follows that \( U_1, \ldots, U_n \) cover \( A \) and we are done.

We have discussed sets a bit, now is the time for functions. We have notions of open, closed, compact sets, and so on, so the idea is to see what reasonable classes of functions, like continuous functions, do to sets of these types.
\( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) function in the usual sense.
\[
\tilde{f}(C) = \{ x \in A : f(x) \in C \}
\]
Compositions, products, etc. are defined in the usual way.
If \( f : A \rightarrow \mathbb{R}^m \), it determines \( m \) component functions \((f_1, \ldots, f_m)\).
\[
f(x) = (f_1(x), \ldots, f_m(x))
\]
Let \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) identity function.
Define \( \pi_i(x) = x_i \) projection function.
We say that \( \lim_{x \to a} f(x) = b \) means that for every \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( |f(x) - b| < \varepsilon \) whenever \( |x - a| < \delta \).
We say that \( f(x) \) is continuous if \( \lim_{x \to a} f(x) = f(a) \) at \( x = a \).

With all the basic definitions in play, we relate open sets and continuity.
Theorem: If $A \subseteq \mathbb{R}^n$, a function $f: A \to \mathbb{R}^m$ is continuous if for every open set $U \subseteq \mathbb{R}^m$ there is an open set $V \subseteq \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$.

Proof: Suppose $f$ is continuous. If $a \in f^{-1}(U)$,

$f(a) \in U \Rightarrow f(a) \in B \subseteq A$.

By continuity, we can ensure that $f(x) \in B$ if $x \in$ small rectangle containing $a$.

Do this for each $a \in f^{-1}(U)$ and let $V$ be the union of all the resulting $C$'s.

It is clear that $f^{-1}(U) = V \cap A$.

Now suppose that for every $U$ open, $f^{-1}(U) = V \cap A$ for some open $V$.

Let $\varepsilon > 0$ be given. The condition $|f(x) - b| < \varepsilon$ means that $f(x) \in$ open rectangle $Ub$, i.e. $f^{-1}(Ub) \cap A$. By assumption, $f^{-1}(Ub)$ is open. This open set contains an open rectangle containing $x$. This can be described in the form $|x - a| < \delta$ and we are done.