Theorem: The vector space $L^\infty$ with norm $\|f\|_{\infty}$ is a complete vector space.

Proposition: $f \in L^\infty$ supported on a set of finite measure.

Then $f \in L^p$ for $p < \infty$, and

$$\lim_{p \to \infty} \|f\|_p = \|f\|_{\infty}, \quad p \to \infty.$$  

Proof: If $\mu(E) = 0$, nothing to prove.

Otherwise,

$$\|f\|_p^p = \left( \int_E |f(x)|^p \, d\mu \right)^{\frac{1}{p}} \leq \left( \int_E \|f\|_\infty^p \, d\mu \right)^{\frac{1}{p}} \leq \|f\|_\infty \mu(E)^{\frac{1}{p}} \to \|f\|_{\infty}$$

So $\lim \|f\|_p = \|f\|_{\infty}$

On the other hand,

$$\mu \left\{ x : |f(x)| \geq \|f\|_{\infty} - \epsilon \right\} \geq \delta$$

for some $\delta > 0$

$$\implies \int_E |f(x)|^p \, d\mu \geq \delta (\|f\|_{\infty} - \epsilon)^p \times \epsilon \lim \|f\|_p = \|f\|_{\infty} - \epsilon$$

ie

$$\lim \|f\|_p \geq \|f\|_{\infty} - \epsilon$$

and we are done.
Banach spaces: complete normed vector spaces

\[ \|v\| = 0 \iff v = 0 \]

\[ \|\alpha v\| = |\alpha| \|v\|, \alpha \text{ scalar, } v \in V \]

\[ \|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V \]

norm axioms

Complete means that if \( \|v_n - v_m\| \to 0 \), \( m \to \infty \), then \( \exists F \in V \) with \( \|v - F\| \to 0 \) as \( m \to \infty \).

Examples: \( L^p(\mathbb{R}^n), 1 \leq p \leq \infty \)

not true below 1 - almost any example will do.

\( C(\mathbb{R}) \)

compact set, \( \mathbb{R} \) norm \( \sup_{x \in X} |f(x)| \)

continuously

functions

Hölder of order \( \alpha \) (\( L^{\alpha}(\mathbb{R}^d) \))

\[ \|f\|_{L^{\alpha}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \]

The case \( \alpha > 1 \) is trivial (check!)
More examples: \( f \in L^p(\mathbb{R}^d) \) has weak derivatives in \( L^p \)
up to order \( k \) if for every \((\alpha_1, \alpha_2, \ldots, \alpha_d)\) with
\[ |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d \leq k \]
\( \exists g_{\alpha} \in L^p \) mixed
partials
\[ \int_{\mathbb{R}^d} g_{\alpha}(x) \phi(x) \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} f(x) \partial_{\alpha} \phi(x) \, dx \]
\( 1 \leq p \leq \infty \)

A smooth functions \( u \) with compact support.

The resulting space is \( L^p_k(\mathbb{R}^d) \) is a Banach space

\( \| \phi \|_{L^p_k(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \| \partial_{\alpha} \phi \|_p \)

by setting \( g_{\alpha} = \partial_{\alpha} \phi \)

In the case \( p = 2 \), we can say more. This will be addressed later.

**Linear functionals and duals:**

\( \mathcal{B} = \) Banach space \( \| \cdot \| \)

\[ \ell: \mathcal{B} \to \mathbb{R}, \quad \ell(\alpha f + \beta g) = \alpha \ell(f) + \beta \ell(g) \]

linear functional scalar scalar \( f, g \in \mathcal{B} \)

\( \ell \) is continuous if \( \| \ell(f) - \ell(g) \| \leq \varepsilon \) whenever
\( \| f - g \| < \delta. \)
$\ell$ (linear functional) is bounded if $\exists M > 0 \ni \|\ell (g)\| \leq M \|g\| \ \forall g \in B.$

**Proposition:** $\ell$ is continuous iff $\ell$ is bounded.

**Linear functional on a Banach space**

**Proof:** Suppose that $\ell$ is continuous. Then $|\ell (g)| < 1$ whenever $\|g\| < \delta$ (for some $\delta > 0$)

$$|\ell \left( \frac{g}{\|g\|} \right) | = \frac{\|\ell (g)\|}{\|g\|} < 1 \quad \text{since} \quad \frac{\|\ell (g)\|}{\|g\|} = \delta$$

It follows that $|\ell (g)| < \frac{1}{\delta} \|g\| \implies \ell$ is bounded

Now suppose that $\ell$ is bounded. Then $|\ell (g)| \leq M \|g\| \implies \ell$ is continuous at the origin.

But $|\ell (g) - \ell (h)| = |\ell (g - h)| \leq M \|g - h\|$, so $\ell$ is continuous everywhere.
The set of all continuous linear functionals on $\mathcal{B}$ is a vector space (easy). The norm is given by

$$
\|f\| = \sup_{\|g\| = 1} |f(g)| = \sup_{\|g\| = 1} |l(g)|
$$

$$
= \sup_{\|g\| = 1} \frac{|l(g)|}{\|g\|}
$$

Theorem: The vector space $\mathcal{B}^*$ is a Banach space.

Proof: The only issue is completeness.

Suppose that $\{l_n\}$ is a Cauchy sequence in $\mathcal{B}^*$. Then $l_n(g)$ is Cauchy, so it converges to $l(g)$.

The mapping $l \mapsto l(g)$ is linear. If $M$ is such that $\|l_n\| \leq M$ for all $n$, we see that

$$
|l(g)| \leq |(l-l_n)g + l_n(g)|
$$

$$
\leq |(l-l_n)g| + M\|g\| \quad \Rightarrow \quad |l(g)| \leq M\|g\|
$$

$\Rightarrow \quad l$ is bounded

It remains to show that $l$ is the limit of $\{l_n\}$.

Given $\varepsilon > 0$, choose $N$ such that $\|l_n - l_m\| < \frac{\varepsilon}{M} \quad \forall n, m > N$. Then if $n > N$, we see that $\forall m > N$ and any $g$,

$$
|l(g)| \leq |(l-l_m)g| + |l_m(g)| \leq |l-l_m|\|g\| + \frac{\varepsilon}{M}\|g\|
$$

$$
\leq \frac{\varepsilon}{M}\|g\| + |l_m(g)| \leq |l_m(g)| + \frac{\varepsilon}{M}\|g\|
$$
If \( m \) is large enough (possibly depending on \( g \)),
\[
(\ell - \ell_m)g \leq \varepsilon \|g\|_2,
\]
so for \( n \geq N \)
\[
(\ell - \ell_n)g \leq \varepsilon \|g\|_2 \quad \text{and we are done.}
\]

We will be able to classify dual Banach spaces in some situations.

If \( \ell = \ell^0 \), \( \ell = \ell^0 \) by elementary linear algebra.

Things get a bit harder when \( \ell = L^p, 1 \leq p < \infty \).

Let \( \frac{1}{p} + \frac{1}{q} = 1 \). Given \( g \in L^q \), consider
\[
e(g) = \int g(x)q(x)dx, \quad \text{a bounded linear functional on } L^p
\]
\[
C \rightarrow L^q \quad \text{when } 1 \leq p < \infty
\]

When \( 1 \leq p < \infty \), we can say more.

**Theorem:** Suppose \( 1 \leq p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, with \( \ell = L^p \), we have \( \ell^* = L^q \). More precisely, for every linear functional \( \ell \) on \( L^p \) there is a unique \( g \in L^q \) so that
\[
e(g) = \int g(x)q(x)dx, \quad \forall \ell \in L^p
\]
Moreover, \( \|\ell\|_{\ell^*} = \|g\|_{L^q} \).