An introduction to the MANDELBROT set, II

JUAN RIVERA-LETELIER

U. of Rochester

Dynamical Systems Workgroup
May 21, 2021

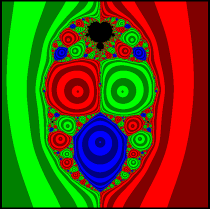


Figure: Failiure of NEWTON's root finding algorithm, from Lukas GEYER website.

Quadratic polynomials: For $c \in \mathbb{C}$, $f_c(z) := z^2 + c$;

Viewed as a dynamical system acting on $\mathbb{C}$.

n-th iterate of f_c: $f_c^n := \underbrace{f_c \circ \cdots \circ f_c}_{n}$;

MANDELBROT set: $\mathcal{M} := \{c \in \mathbb{C} : (f_c^n(0))_{n=1}^{\infty} \text{ is bounded}\}$.

$\sim$

Periodic point of period n: $p \in \mathbb{C}$ such that $f_c^n(p) = p$.

- Orbit: $O(p) := \{p, f_c(p), \ldots, f_c^{n-1}(p)\}$;
- Multiplier: $|Df_c^n(p)|$;

  Same for every periodic point in $O(p)$.

- p is:
  - attracting if $|Df_c^n(p)| < 1$;
  - indifferent if $|Df_c^n(p)| = 1$;
  - repelling if $|Df_c^n(p)| > 1$;

$\mathcal{H} := \{c \in \mathbb{C} : f_c \text{ has an attracting periodic point}\} \subset \mathcal{M}$.

By FATOU's theorem;
Hyperbolic component of $\mathcal{M}$: Connected component of $\mathcal{H}$.
For $c \in \mathcal{H}$, SMALE's hyperbolicity theory applies to f_c.

FATOU Conjecture
$\mathcal{H}$ is dense in $\mathcal{M}$.

$\Rightarrow$ Hyperbolicity is dense in the quadratic family;
It is known to be false in higher dimensions.

MLC Conjecture (DOUADY–HUBBARD)
The MANDELBROT set is locally connected.

The MANDELBROT set is connected (DOUADY–HUBBARD, SIBONY).

Theorem (DOUADY–HUBBARD)
MLC Conjecture $\Rightarrow$ FATOU conjecture.

$\Rightarrow$ THURSTON's combinatorial model of the MANDELBROT set is accurate.

Plan

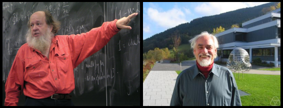


Figure: Adrian Douady and John Hubbard.

∼

Plan:

1. Hyperbolic components attached to the main cardioid;
2. External rays and the limbs of the Mandelbrot set;
3. Yoccoz' inequality and MLC at points of the main cardioid.

Review of Part I

- $\mathscr{M} \cap \mathbb{R} = [-2, \frac{1}{4}]$. For $c \in [-2, \frac{1}{4}]$:
  - *Fixed points of* f_c: $\alpha(c) := \frac{1-\sqrt{1-4c}}{2}, \beta(c) := \frac{1+\sqrt{1-4c}}{2}$;
  - *Invariant interval:* $I(c) := [-\beta(c), \beta(c)]$, $f_c(I(c)) \subseteq I(c)$.

Up to a change of coordinates,
$f_c|_{I(c)}$ is the logistic map $g_\lambda(x) := \lambda x(1-x)$ with $\lambda = 1 + \sqrt{1-4c}$.

- Main hyperbolic component or hyperbolic component of period one:

$$W_1 := \{c \in \mathbb{C} : f_c \text{ has an attracting fixed point}\}$$
$$= \left\{\frac{\lambda}{2} - \frac{\lambda^2}{4} : \lambda \in \mathbb{D}\right\};$$

- Hyperbolic component of period two:

$$W_2 := \{c \in \mathbb{C} : f_c \text{ has an attracting periodic point of minimal period } 2\}$$
$$= \left\{\frac{\lambda - 1}{4} : \lambda \in \mathbb{D}\right\}.$$

Period doubling bifurcation

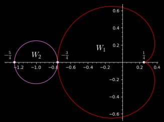


Figure: Hyperbolic components of periods 1 and 2.

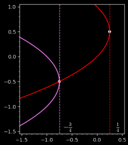


Figure: Period doubling bifurcation

- At $c = -\frac{3}{4}$: $Df_c(\alpha(c)) = -1 \Rightarrow \alpha(c)$ is indifferent;
- For $c < -\frac{3}{4}$: $\alpha(c)$ is repelling & orbit of period 2 appears.

Period doubling bifurcation: For every generic family $(g_\lambda)_\lambda$ such that;
$g_{\lambda_*}(p_*) = p_*$ and $Dg_{\lambda_*}(p_*) = -1$;
Period doubling movie.

Period doubling bifurcation

$$\begin{aligned} \mu : \mathbb{C} &\to \mathbb{C} \\ \lambda &\mapsto \mu(\lambda) := \frac{\lambda}{2} - \frac{\lambda^2}{4}. \end{aligned}$$

Parametrization of W_1;
$\mu(\lambda)$: unique c such that f_c has a fixed point $\alpha(c)$ of multiplier λ.

For θ in $\mathbb{R}$: Internal ray of angle $\theta = \mu(\{r \exp(2\pi i\theta) : r \in [0,1)\})$.

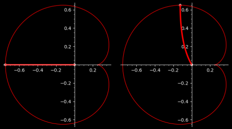


Figure: Internal ray of angles $\frac{1}{2}$ and $\frac{1}{3}$.

Period tripling bifurcation

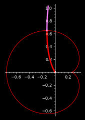


Figure: Internal ray of angle $\frac{1}{3}$, extended.

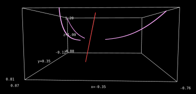
For $r > 0$:

- $c(r) := \mu(r \exp(2\pi i \frac{1}{3}))$;
- $\alpha(r)$: Fixed point of $f_{c(r)}$ of multiplier $r \exp(2\pi i \frac{1}{3})$.

At $r = 1$: $Df_{c(1)}(\alpha(1)) = \exp(2\pi i \frac{1}{3})$

$\Rightarrow \alpha(1)$ is indifferent;

Period tripling bifurcation



For $r > 1$: $\alpha(r)$ is repelling & orbit of period 3 "appears".

Period tripling movie.

For $r > 1$ close to 1: The new periodic orbit is attracting

$\Rightarrow c(r) \in$ hyperbolic component of period 3.

Period tripling bifurcation: For every generic family $(g_\lambda)_\lambda$ such that;
$g_{\lambda_*}(p_*) = p_*$ and $Dg_{\lambda_*}(p_*) = \exp(2\pi i \frac{1}{3})$;
FATOU's theorem $\Rightarrow$ genericity condition.

Period multiplying bifurcation

For every rational number $\frac{p}{q}$ in $(0, 1)$:

- $c(r) := \mu(r \exp(2\pi i \frac{p}{q}))$;
- $\alpha(r)$: Fixed point of $f_{c(r)}$ of multiplier $r \exp(2\pi i \frac{p}{q})$.
- At $r = 1$: $Df_{c(1)}(\alpha(1)) = \exp(2\pi i \frac{p}{q}) \Rightarrow \alpha(1)$ is indifferent;
- For $r > 1$: $\alpha(r)$ is repelling & orbit of period q "appears";
- For $r > 1$ close to 1: The new periodic orbit is attracting

$\Rightarrow c(r) \in$ hyperbolic component of period q.

$\rho_{\frac{p}{q}} := \mu(\exp(2\pi i \frac{p}{q}))$;
$H_{\frac{p}{q}}$: Hyperbolic component containing $c(r)$, for $r > 1$ close to 1.

$H_{\frac{1}{2}} = W_2$;
$\partial H_{\frac{p}{q}}$ is tangent to ∂W_1 at $\rho_{\frac{p}{q}}$;

External rays

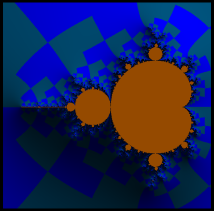


Figure: MANDELBROT set, from Tomoki KAWAHIRA's gallery.

Theorem (Douady–Hubbard)
There is a unique conformal map

$$\Phi\colon \mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\mathcal{M}$$

that is tangent to the identity at ∞.

By Carathéodory's theorem:
MLC $\Rightarrow$ Φ extends continuously to $\partial\mathbb{D}$.

Definition
For θ in $\mathbb{R}$: The external ray of angle θ of $\mathcal{M}$, is

$$\mathcal{R}(\theta) := \{\Phi(r\exp(2\pi i\theta)) : r > 1\}.$$

If

$$\lim_{r\to 1^+} \Phi(r\exp(2\pi i\theta))$$

exists, then $\mathcal{R}(\theta)$ lands and the limit is the landing point of $\mathcal{R}(\theta)$.

External rays

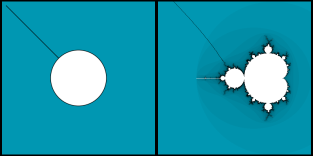
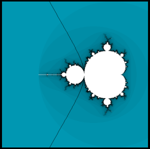


Figure: External ray of angle $\frac{3}{8}$.

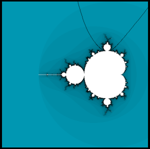
External rays

Theorem (Douady–Hubbard)
For $\frac{p}{q}$ *in* $(0,1)$: $\rho_{\frac{p}{q}}$ *is the landing point of precisely two rays.*

$\theta^-\left(\frac{p}{q}\right)$, $\theta^+\left(\frac{p}{q}\right)$: Angles of rays landing at $\rho_{\frac{p}{q}}$.



$\theta^-\left(\frac{1}{2}\right) = \frac{1}{3}$, $\theta^+\left(\frac{1}{2}\right) = \frac{2}{3}$.



$\theta^-\left(\frac{1}{3}\right) = \frac{1}{7}$, $\theta^+\left(\frac{1}{3}\right) = \frac{2}{7}$.

External rays

For $\frac{p}{q}$ in $(0,1)$:

$$\mathcal{R}\left(\theta^-\left(\tfrac{p}{q}\right)\right) \cup \{\rho_{\frac{p}{q}}\} \cup \mathcal{R}\left(\theta^+\left(\tfrac{p}{q}\right)\right)$$

cuts the plane in 2 parts.

$W_{\frac{p}{q}}$: Piece containing $H_{\frac{p}{q}}$.

$\frac{p}{q}$-Wake.

$L_{\frac{p}{q}} := \mathcal{M} \cap W_{\frac{p}{q}}$.

$\frac{p}{q}$-Limb of $\mathcal{M}$;
$L_{\frac{p}{q}}$ contains a copy of $\mathcal{M}$.

Theorem
$\mathcal{M} = \overline{W_1} \cup \left(\bigcup_{\frac{p}{q}\in(0,1)\cap\mathbb{Q}} L_{\frac{p}{q}}\right).$

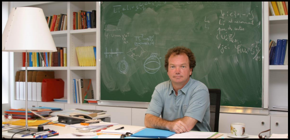


Figure: Jean-Chiristophe Yoccoz.

Theorem (Yoccoz)

The Mandelbrot set is locally connected at every point of ∂W_1.

Yoccoz' inequality

$\frac{p}{q}$: *Rational number in* $(0,1)$;

$r(q) := \frac{\log 2}{q}$.

In the coordinate $\log \lambda$, *we have*

$$L_{\frac{p}{q}} \subset B\left(2\pi i \frac{p}{q} + r(q), r(q)\right).$$

Disc of radius $r(q)$ tangent to the imaginary axis at $2\pi i \frac{p}{q}$;

Roughly: The diameter of $L_{\frac{p}{q}}$ is $\leq \frac{\text{Const.}}{q}$;

$\Rightarrow$ MLC at every point of ∂W_1;

Idea of proof.