The Real Fatou Conjecture.

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Fatou's Conjecture:

Hyperbolic maps are open and dense among all rational maps.

Recall (hyperbolic maps).

A rational map $f: \mathbb{C} \longrightarrow \mathbb{C}$, of deg $d > 1$, is said to be hyperbolic if all critical points of tend to an attracting cycle.

History:

- 1920 - Fatou's memoir.
- 1971 - Jakobson. Solution in the C^1 topology.
- 1992, Sullivan.
- 1997, Graczyk and Swiatek, and Lyubich. Solution in the real quadratic case.
- 2003, Kozlovski. Solution in the smooth unimodal case.
- 2005, Kozlovski, Shen, van Strien. Real polynomials with real crit. points
- 2004, ——, Real polynomial.

Quasiconformal Maps. $\frac{\partial}{\partial z} = \frac{1}{2i}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

- $\mathcal{U}, V \subset \mathbb{C}$ non-empty, open. $\frac{\partial}{\partial \bar{z}} = \frac{1}{2i}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$
- $f: \mathcal{U} \longrightarrow V$ orientation-preserving hom.

f is called quasiconformal if:

① f is absolutely continuous on line.

For any $[a,b]\times[c,d] \subset \mathcal{U}$

$$\left.\begin{array}{l} x \longmapsto f(x+iy) \\ y \longmapsto f(x+iy) \end{array}\right\} \text{are absolutely continuous.}$$

② $\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right|$ a.e. in $\mathcal{U}$ for some $0 \leq k < 1$.

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* f is K-quasiconformal with $K := \frac{1+k}{1-k} \geq 1$.

Proposition:

$f : \mathcal{U} \longrightarrow V$ is K-quasiconformal $(K \geq 1)$ iff for all annuli $A \subseteq \mathcal{U}$

$$\frac{1}{K} \operatorname{mod}(A) \leq \operatorname{mod}(f(A)) \leq K \operatorname{mod}(A).$$

Definition:

Let $\mathcal{U}$ and V be non-empty open sets in $\mathbb{C}$. An orientation preserving homeomorphism $f : \mathcal{U} \longrightarrow V$ is called quasiconformal if:

① f is absolutely continuous on line.

For any $[a,b] \times [c,d] \subset \mathcal{U}$

$$\left.\begin{array}{l} x \longmapsto f(x+iy) \\ y \longmapsto f(x+iy) \end{array}\right\} \text{ are absolutely continuous.}$$

The Measurable Riemann Mapping Theorem.

Let μ be a measurable Beltrami coeff. on the Riemann sphere $\hat{\mathbb{C}}$ with $\|\mu\|_\infty = k < 1$. Then

① There exists a unique quasiconformal homeomorphism $f = f^{[\mu]} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ which fixes $0, 1, \infty$ and solves the Beltrami equation

$$K = \frac{1+|\mu|}{1-|\mu|} \qquad \mu \cdot \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar{z}} . \qquad z = x+iy$$

$$\gamma(x,y)|dz| \rightsquigarrow \gamma(x,y)|dz + \mu d\bar{z}|$$

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② f is a C^∞ (resp. real analytic) diffeomorphism if μ is C^∞ (resp. real analytic).

③ The family f_t given by ① for the Beltrami form $t\mu$, depends holomorphically on the complex parameter $t \in B(0, 1/k)$.

The Dense Hyperbolicity Theorem [Graczyk and Swiatek 97, Lyubich 97]

In the real quadratic family

$$f_a(x) = ax(1-x) \quad ; \quad 0 \leq a \leq 4$$

the mapping f_a has an attracting cycle, and thus is hyperbolic, for an open and dense set of parameters a.

Main Theorem:

- $f, \hat{f} \longrightarrow$ real quadratic polynomials.
  bounded foward critical orbit.
  no attracting or indiferent cycle.

Then, if they are $\overbrace{\text{topologically conjugated}}^{\text{in } \mathbb{R}}$, the conjugacy extends to a quasiconformal conjugacy between their analytic continuations to the complex plane.

Main Theorem $\Rightarrow$ Dense Hyperbolicity.

Fact 1:

$C_a := \{ b \in [0,4] : f_a \text{ and } f_b \text{ are q.c.-conj. on } \mathbb{C} \}$.

Then $C_a = \{a\}$ or is open.

Proof:

- f_a and f_b q.c.-conj, $a \neq b$.
- H q.c. conjugacy between f_a and f_b.

$\lambda = \gamma(x,y)\,|dz + \mu\, d\bar{z}|$ $\quad f^*\lambda \rightsquigarrow f^*\mu$

Since H is q.c. there is an f_b-invariant Beltrami coefficient μ not identically 0.

- $c \in \mathbb{C}$; $|c| < \|\mu\|_\infty^{-1}$.
- H_c given by ③ in the MRMT.

Then H_c dependes analytically on c and

$$f_{r(c)} = H_c \circ f_b \circ H_c^{-1}$$

is a analytic family of analytic functions. ($f_{r(c)}$ is 1-q.c.) and is a 2-1 branched covering of the Riemann sphere fixing 0 and ∞.

φ q.c. $\longrightarrow \mu$ Beltrami coefficient.

$\mu \leftarrow \varphi^* (\gamma(x,y) |dz|)$

$\mu = \varphi^* \mu_0 \quad \rightsquigarrow \quad \mu = \dfrac{\varphi_{\bar z}}{\varphi_z}$

$$f_b^* \mu = f_b^* (\varphi^* \mu_0)$$

$$= (\varphi \circ f_b)^* \mu_0$$

$$= (f_a \circ \varphi)^* \mu_0$$

$$= \varphi^* (\underbrace{f_a^* \mu_0})$$

$$= \varphi^* \mu_0 = \mu.$$

Since ∞ is also a branching point, $f_{r(c)}$ is a family of quadratic polynomials.

- $e(c) := \frac{d}{dz} f_{r(c)}(0)$ is analytic
- $e(\{ |c| < \|\mu\|_\infty \})$ is a point or an open set.

Since $r(1) = a$ and $r(0) = b$ $e(\{ |c| < \|\mu\|_\infty \})$ is an open set.

$\boxtimes$

Fact 2:

There are only countably many complex values of a for which the map $x \longmapsto ax(1-x)$ has a neutral periodic point.

Proof:

- $k > 0$, $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$

Then, the pair of eq.

$$f_a^k(z) = z \quad \text{and} \quad \frac{d f_a^k}{dz}(z) = \lambda$$

has only finitely many solutions (a, z).

Theorem:

- $P(a,z)$ irreducible polynomial of two complex variables.

Then $\overbrace{\{\text{solutions of } P(a,z)=0\} \cup \{\infty\}}^{V}$ has the structure of a compact Riemann surface. Moreover, the projections are meromorphics of this surface.

Then the set of solutions of $f_a^k(z) - z = 0$ splits into the union of finitely many compact Riemann surfaces.

- $\dfrac{d f_a^k}{dz}(z)$ is meromorphic on each RS.

If $\dfrac{d f_a^k}{dz}(z) = \lambda$ infinitely times, by the identity principle it must be constant in one of the surfaces, say S.

$\hat{S} := \{(a,z) \text{ solving both equations}\} \subset V$

Then, $\pi_1(\hat{S}) \subseteq$ connectedness locus

$\pi_2(\hat{S}) \subseteq$ filled Julia set. f_a

Since $\# \hat{S} < +\infty$; $\pi_1(\hat{S})$ and $\pi_2(\hat{S})$ are compact (π_1 and π_2 are continous).

Also π_1, π_2 meromorphic $\Rightarrow$ open or constant.

$\Rightarrow$ $\pi_1(\hat{S})$ and $\pi_2(\hat{S})$ is a single point.

$\boxtimes$

Fact 3 [Guckenheimer 79]

- f, g real quadratic polynomials.
  bounded critical orbit.
  f has no attracting periodic orbits.

- z_f, z_g := critical points of f and respectively
  (= maximal points)

f and g are topologically conjugate iff for every $n > 0$

$$\operatorname{sgn}(f^n(z_f) - z_f) = \operatorname{sgn}(g^n(z_g) - z_g)$$

- $f_a(x) = ax(1-x)$ has no stable periodic orbits.
- $T_a := \{ b : f_a$ and f_b are top. conjugated in $\mathbb{R} \}$

Claim: T_a is closed.

Proof:

- $\{b_n\}_{n \geq 1} \subseteq T_a$, $b_n \longrightarrow b$.

$\Rightarrow$ $\operatorname{sgn}\left(f_{b_n}^k(1/2) - 1/2\right)$ remain fixed for all n.

By continuity $\operatorname{sgn}\left(f_b^k(1/2) - 1/2\right)$ remain the same (claim follows from Fact 3),

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Or $\operatorname{sgn}\left(f_b^k(1/2) - 1/2\right) = 0$ for some k.

$\Rightarrow$ $1/2$ periodic by f_b $\Rightarrow$ $\forall\, \alpha \in B(b, \delta)$ f_α has an attracting cycle $\times$ $\boxtimes$

Fact 1 : $a \in (0,4]$ s.t. f_a has only repelling periodic orbits $\Rightarrow$ $C_a \cap \mathbb{R} = T_a$ is either open or a point. Since T_a is closed, it must be a point.

By Fact 2 we can consider $a \in (0,4]$ s.t f_a has only repelling periodic orbits.
$a_1 \neq a$ s.t. f_{a_1} has only repelling periodic orbits.
$\Rightarrow$ f_a and f_{a_1} are not top. conj.
By Fact 3 for some.

$$\operatorname{sgn}\left(f_a^k(1/2) - 1/2\right) \neq \operatorname{Sgn}\left(f_{a_1}^k(1/2) - 1/2\right).$$

By the intermediate value theorem, exists a_0 between a_1 and a s.t.

$$f_{a_0}^k(1/2) - 1/2 = 0.$$

So f_{a_0} is hyperbolic.

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- John Guckenheimer. Sensitive Dependence to Initial Conditions for One-Dimensional Maps. Comm. Math. Phys. 1979.