Matir 161: Calculus IA
Second Midterm Exam
November 18, 2010

NAME (please print legibly): KEY
Your University ID Number: ______________________________________________________
Indicate your instructor with a check in the box:

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Schedule</th>
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<tbody>
<tr>
<td>Amanda Beeson</td>
<td>MWF 9:00 - 9:50 AM</td>
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<tr>
<td>Amanda Beeson</td>
<td>MWF 10:00 - 10:50 AM</td>
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<tr>
<td>Nsoki Mavinga</td>
<td>MWF 11:00 - 11:50 AM</td>
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<tr>
<td>Steve Lester</td>
<td>T, Th 2:00-3:15 PM</td>
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- The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.
- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Put your answers in the spaces provided.
- You are responsible for checking that this exam has all 12 pages.

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1. (8 pts) The position of a particle at time $t$ is given by

$$s(t) = t^2 - 4t.$$

a) Find the velocity at time $t$.

$$v(t) = s'(t) = 2t - 4$$

b) When is the particle moving forward?

The particle is moving forward when $v(t) > 0$, i.e.,

$$2t - 4 > 0 \Rightarrow t > 2$$

c) Find the total distance traveled by the particle during the first five seconds.

The particle is travelling backwards for $0 < t < 2$ and forwards for $2 < t \leq 5$. So, the distance travelled is

$$|s(2) - s(0)| + |s(5) - s(2)| = |-4 - 0| + |5 - 4|$$

$$= 4 + 1 = 13.$$
2. (7 pts) Two ships start at the same location. One travels north at 10 km/h the other travels west at 20 km/h. Find the rate at which the distance between the ships is changing after two hours.

Given: \[ \frac{dx}{dt} = 20 \text{ km/h} \quad \frac{dy}{dt} = 10 \text{ km/h} \]

Want: \[ \frac{dD}{dt} \] after \( t = 2 \) hours.

Know: \[ D^2 = x^2 + y^2 \]

So \[ 20 dD \frac{dt}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \]

\[ 2 (20\sqrt{5}) \frac{dD}{dt} = 2 (20)(10) + 2 (40)(20) = 400 + 1600 = 2000 \]

\[ \Rightarrow \frac{dD}{dt} = \frac{50}{\sqrt{5}} = 10\sqrt{5} \]
3. (10 pts) Find the derivative $\frac{dy}{dx}$ of each of the following functions.

a) $y = \ln(e^x \cos x)$

First simplify.

$$\ln(e^x \cos x) = \ln(e^x) + \ln(\cos x) = x \ln(e) + \ln(\cos x) = x + \ln(\cos x) \quad (\ln(e) = 1)$$

$$\frac{dy}{dx} = \frac{d}{dx} (x + \ln(\cos x)) = 1 + \frac{1}{\cos x} \cdot \frac{d}{dx} \cos x = 1 - \frac{\sin(x)}{\cos(x)} = 1 - \tan(x)$$

b) $y = x^{\sin x}$

Solution 1

First, find the logarithmic derivative of $y$.

$$\ln(y) = \ln(x^{\sin x}) = \sin(x) \ln(x)$$

Now differentiate.

$$\frac{1}{y} \frac{dy}{dx} = \frac{\sin(x)}{x} + \cos(x) \ln(x) \Rightarrow \frac{dy}{dx} = y \left( \frac{\sin(x)}{x} + \cos(x) \ln(x) \right)$$

Solution 2

$$y' = x^{\sin(x)} \ln(x) \quad (\sin(x) \ln(x))' = x^{\sin(x)} \left( \frac{\sin(x)}{x} + \cos(x) \ln(x) \right)$$
4. (15 pts) Evaluate the following limits. If it does not exist write DNE. If it is infinite, determine whether it is $+\infty$ or $-\infty$. Explain your reasoning.

(a) \( \lim_{x \to 0^+} (1 + 3x)^{\frac{1}{x}} \)

Notice \( (1+3x)^{\frac{1}{x}} = e^{\ln (1+3x) \cdot \frac{1}{x}} \).

Now \( \lim_{x \to 0^+} \ln (1+3x) = \ln (1+3 \cdot 0) = \ln (1) = 0 \) and \( \lim_{x \to 0^+} \frac{1}{x} = \infty \).

So \( \lim_{x \to 0^+} \frac{\ln (1+3x)}{x} = 0 \) and \( \lim_{x \to 0^+} \frac{1}{x} = \infty \).

Apply L'Hopital's rule to get

\[ \lim_{x \to 0^+} \frac{\ln (1+3x)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{1+3x} \cdot 3}{1} = \frac{3}{1} = 3 \]

(b) \( \lim_{x \to +\infty} \frac{e^x + x}{e^{2x}} \)

\[ \lim_{x \to +\infty} e^x + x = +\infty, \quad \lim_{x \to +\infty} e^{2x} = +\infty \]

So, apply L'Hopital's rule.

\[ \lim_{x \to +\infty} \frac{e^x + 1}{2e^{2x}} = \lim_{x \to +\infty} \frac{e^x}{4e^{2x}} = \lim_{x \to +\infty} \frac{1}{4e^x} = 0 \]

(c) If \( f'(x) \) is continuous, \( f(5) = 0 \) and \( f'(5) = -5 \), evaluate

\[ \lim_{x \to 0} \frac{f(5+2x) + f(5+3x)}{x} \]

Since \( f \) is differentiable, \( f \) is continuous.

So \( \lim_{x \to 0} f(5+2x) + f(5+3x) = f(5) + f(5) = f(5) + f(5) = 0 \)

\[ \lim_{x \to 0} x = 0 \]

So, applying L'Hopital's rule yields

\[ \lim_{x \to 0} \frac{f'(5+2x) \cdot 2 + f'(5+3x) \cdot 3}{1} = 2f'(5) + 3f'(5) \]

\[ = -5 \cdot 2 + (-5) \cdot 3 \]

\[ = -10 - 15 = -25 \]
5. (10 pts)

(a) Find the linearization of the function $f(x) = \sqrt{1-x}$ at $x = 0$.

The linearization of a differentiable function at $x = a$ is defined as

$$L(x) = f(a) + f'(a)(x-a),$$

where

$$f(0) = \sqrt{1-0} = 1,$$

$$f'(x) = -\frac{1}{2} (1-x)^{-3/4},$$

$$f'(0) = -\frac{1}{4}.$$

Therefore,

$$L(x) = 1 - \frac{1}{4} (x-0) = 1 - \frac{x}{4}.$$

(b) Use this linearization to approximate the number $\sqrt{0.9}$.

$$f\left(\frac{1}{10}\right) = \frac{\sqrt{0.9}}{10},$$

so

$$L\left(\frac{1}{10}\right) \approx \frac{\sqrt{0.9}}{10}.$$
6. (6 pts)

Let \( f(x) = x^2 - 4x \). Find the absolute maximum and minimum values of \( f(x) \) on the closed interval \([0, 5]\).

1. First find critical values
   \[
   f'(x) = 2x - 4
   \]
   set \( f'(x) = 0 \), so \( 2x - 4 = 0 \) \( \Rightarrow \) \( x = 2 \).

2. Evaluate \( f \) at critical values
   \[
   f(2) = 4 - 4(2) = -4
   \]

3. Evaluate \( f \) at the endpoints of \([0, 5]\)
   \[
   f(0) = 0, \quad f(5) = 5
   \]

   **Maximum value of \( f \) on \([0, 5]\):** 5 and occurs at \( x = 5 \).

   **Minimum value of \( f \) on \([0, 5]\):** -4 and occurs at \( x = 2 \).
7. (10 pts) Let \( f(x) \) be the function
\[ x^3 - 2x \]
with domain \([0, 1]\).

(a) Compute the average slope of \( f(x) \) on the interval \([0, 1]\).

\[
\text{Average slope of a cont. function } f \text{ on } [a,b] = \frac{f(b) - f(a)}{b - a}
\]

So \( f(x) = x^3 - 2x \) so
\[
\text{Ave. slope } \approx \frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 0}{1 - 0} = -1
\]

(b) Find the number \( c \) that satisfies the conclusion of the Mean Value Theorem.

The Mean Value Theorem tells us that there is a \( c \in (0, 1) \) so that \( f'(c) = \frac{f(1) - f(0)}{1 - 0} = -1 \).

Now \( f'(x) = 3x^2 - 2 \), so set \( f'(x) = -1 \)
\[
3x^2 - 2 = -1 \quad \Rightarrow \quad x = \frac{\sqrt{3}}{3}, \quad \frac{-\sqrt{3}}{3}
\]

Now \( 0 < c < 1 \) so \( c = \frac{\sqrt{3}}{3} \).
8. (9 pts)

A triangle has vertices at (0,1), (2,0), and (-2,0). Find the dimensions of the rectangle of largest area that can be inscribed within the triangle if one side of the rectangle lies on the base of the triangle.

\[
A = \text{"height"} \cdot \text{"length"} = 2 \times y
\]

**Solution 1:**

The equation of the line is \( y = 1 - \frac{1}{2}x \),

\[
A = 2x(1 - \frac{1}{2}x) = 2x - x^2
\]

\[
A' = 2 - 2x \quad \text{Now set } A' = 0 \quad \text{so } x = 1,
\]

\[
y = 1 - \frac{1}{2}(1) = \frac{1}{2}, \quad \text{and } A = 2(1)(\frac{1}{2}) = 1
\]

**Solution 2:**

by similar triangles

\[
\frac{2-x}{y} = 2 \quad \Rightarrow \quad y = 1 - \frac{1}{2}x \quad \text{as before}
\]

we get \( x = 1, \ y = \frac{1}{2} \) \( \text{Base} = 2, \text{height} = \frac{1}{2} \)
9. (25 pts)

Let \( f(x) = \frac{1-x}{x^2} \). Then \( f'(x) = \frac{x-2}{x^3} \) and \( f''(x) = \frac{2(3-x)}{x^4} \).

a) What is the domain of \( f(x) \)?

\[ D = (-\infty, 0) \cup (0, \infty) \]

b) Find the \( x- \) and \( y- \)intercepts of the graph of \( f(x) \).

\( x- \)int: \( \frac{1-x}{x^2} = 0 \Rightarrow x = 1 \) so \( (1, 0) \)

\( y- \)int: \( f \) is not defined at \( x = 0 \)

so \( f \) has no \( y- \)int.

c) Is \( f(x) \) odd? even? neither?

\[ f(-x) = \frac{1-(-x)}{(-x)^2} = \frac{1+x}{x^2} \]

\( \frac{1+x}{x^2} \neq f(x) \)

\( \frac{1+x}{x^2} \neq -f(x) \) so \( f \) is

neither even nor odd.

d) Find all horizontal and vertical asymptotes of the graph of \( f(x) \).

**Horizontal:** \( \lim_{x \to \infty} \frac{1-x}{x^2} = \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 0 - 0 = 0 \)

\( \lim_{x \to -\infty} \frac{1-x}{x^2} = \lim_{x \to -\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 0 - 0 = 0 \) so \( y = 0 \)

**Vertical:** \( \lim_{x \to 0^+} \frac{1-x}{x^2} = \infty \)

\( \lim_{x \to 0^-} \frac{1-x}{x^2} = \infty \)

So \( x = 0 \).
e) Find all intervals on which \( f(x) \) increases and/or decreases.

\[
\frac{f'(x)}{x^3} = \frac{x-2}{x^3} \quad \text{Set } f'(x) = 0
\]

\[
\frac{x-2}{x^3} = 0 \implies x = 2.
\]

Increasing:
\(-\infty, 0) \cup (2, \infty)\)
Decreasing:
\((0, 2)\)

f) Find all critical numbers for \( f(x) \) and classify any local maxima or minima.

The only critical number is 2.

\( x = 2 \) is a local minimum, by the first derivative test (see part e).

\[
f(2) = \frac{1-2}{2^2} = -\frac{1}{4}
\]

So, \( f \) has a local min. of \(-\frac{1}{4}\) at \( x = 2 \).

g) Find intervals on which the graph of \( f(x) \) is concave up and concave down.

\[
f''(x) = \frac{2(3-x)}{x^4}
\]

Set \( f'(x) = 0 \)

\[
x = 3
\]

Concave up:
\(-\infty, 3) \cup (3, \infty)\)

Concave down:
\((3, \infty)\)
h) Find all points of inflection.

\[ \text{Solution to } (g) \Rightarrow \text{ There is one inflection point at } (3, -\frac{2}{9}). \]

i) Finally, sketch a graph of \( f(x) \) indicating clearly all of the above attributes of the graph.