Math 161: Calculus IA

Final Exam
December 16, 2010

NAME (please print legibly):

Your University ID Number:

Indicate your instructor with a check in the box:

<table>
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<tr>
<th>Instructor</th>
<th>Time</th>
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<tr>
<td>Amanda Beeson</td>
<td>MWF 9:00 - 9:50 AM</td>
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<td>Amanda Beeson</td>
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<td>Nsoki Mavinga</td>
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<td>Steve Lester</td>
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- The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.

- Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

- Put your answers in the spaces provided.

- You are responsible for checking that this exam has all 19 pages.

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PART A.

1. (10 pts) Consider the function

\[ g(x) = \begin{cases} 
  x - 1 & \text{if } x < 4, \\
  5 - x^2 & \text{if } x \geq 4.
\end{cases} \]

a) Evaluate \( \lim_{x \to 4^-} g(x) \).

\[
= \lim_{x \to 4^-} x - 1 = 3
\]

b) Evaluate \( \lim_{x \to 4^+} g(x) \).

\[
= \lim_{x \to 4^+} 5 - x^2 = -11
\]

c) Is this function continuous at \( x = 3 \)? Justify your answer!

\[
\lim_{x \to 3} g(x) = \lim_{x \to 3} x - 1 = 2 = g(3)
\]

thus the function is continuous at 3
2. **(16 pts)** Differentiate the following functions.

a) \( f(x) = 3x^2 + \sqrt{x} \cos(x) \)
\[
\frac{df}{dx} = 6x + \sqrt{x} (-\sin x) + \frac{1}{2} x^{-\frac{1}{2}} \cos x
\]

b) \( f(x) = \frac{\ln(x)}{x} \)
\[
\frac{df}{dx} = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}
\]

c) \( f(x) = \arctan(e^{-x}) \)
\[
\frac{df}{dx} = \frac{1}{1 + (e^{-x})^2} \cdot e^{-x} \cdot (-1) = -\frac{e^{-x}}{1 + e^{-2x}}
\]

d) Find \( \frac{dy}{dx} \) where \( y \) is defined implicitly by \( x^2 + xy + y^2 - 7 = 0 \).

Take \( \frac{d}{dx} \) both sides, we have
\[
2x + x \cdot \frac{dy}{dx} + y + 2y \cdot \frac{dy}{dx} = 0
\]
\[
(x + 2y) \frac{dy}{dx} = -y - 2x
\]
\[
\frac{dy}{dx} = -\frac{y + 2x}{x + 2y}
\]
3. (10 pts) Let $f(x) = x^3 - 3x^2 - 24x + 2$.

   a) Find an equation of the tangent line to the curve $y = f(x)$ at $x = 0$.

   $f'(x) = 3x^2 - 6x - 24$.
   $f'(0) = -24, \quad f(0) = 2$.
   
   Point (0, 2), slope $-24$.

   $f' -2 = -24 \times x$.

   b) Find the point(s) at which $f(x)$ has a horizontal tangent line.

   Horizontal tangent line $\implies f'(x) = 0$.

   $3x^2 - 6x - 4 = 0$
   $x^2 - 2x - 8 = 0$
   $(x-4)(x+2) = 0$

   $\implies x = 4 \quad \text{or} \quad x = -2$.
4. (8 pts) Suppose \( f(x) = \frac{x - 2}{|x - 2|} \).

a) Sketch a graph of \( f(x) \) on \([0, 4]\).

\[
\begin{align*}
f(x) &= \begin{cases} 
1 & x > 2 \\
-1 & x < 2
\end{cases}
\end{align*}
\]

Note: undefined at \( x = 2 \).

b) Now suppose \( F(x) \) is a function with \( F(2) = 0 \) and \( F'(x) = f(x) \) on \([0, 4]\). Sketch a graph of \( F(x) \) on \([0, 4]\).

\[
\begin{align*}
F(x) &= \begin{cases} 
x - 2 & x > 2 \\
2 - x & x \leq 2
\end{cases}
\end{align*}
\]
5. (6 pts) Let \( f(x) = \frac{6x^2 + 2}{(x^2 - 1)^3} \).

a) Find the horizontal asymptote(s), if any. Show your work. (You will not receive full credit if do not use limits to justify your answer.)

\[
\lim_{x \to \infty} \frac{6x^2 + 2}{(x^2 - 1)^3} = \lim_{x \to \infty} \frac{12x}{3(x^2 - 1)^2 \cdot 2x} = \lim_{x \to \infty} \frac{2}{(x^2 - 1)^2} = 0
\]

L'Hopital's Rule

\[
\lim_{x \to -\infty} \frac{6x^2 + 2}{(x^2 - 1)^3} = 0 \\
\text{Similarly, thus } y = 0 \text{ is a horizontal asymptote.}
\]

b) Find the vertical asymptote(s), if any, and describe the behavior of \( f(x) \) near the vertical asymptote(s). Show your work. (You will not receive full credit if do not use limits to justify your answer.)

\( x^2 - 1 = 0 \) \( \Rightarrow \) gives rise to vertical asymptotes.

i.e., \( x = \pm 1 \)

\[
\lim_{x \to 1^+} \frac{6x^2 + 2}{(x^2 - 1)^3} = \infty \\
\lim_{x \to 1^-} \frac{6x^2 + 2}{(x^2 - 1)^3} = -\infty
\]

\[
\lim_{x \to -1^+} \frac{6x^2 + 2}{(x^2 - 1)^3} = -\infty \\
\lim_{x \to -1^-} \frac{6x^2 + 2}{(x^2 - 1)^3} = \infty
\]
PART B.

6. (20 pts) Let \( f(x) = 6x^5 - 10x^3 \).

a) Find the \( x \)- and \( y \)-intercepts of the graph of \( f(x) \).

**\( x \)-intercepts:**
\[
0 = 6x^5 - 10x^3 = \frac{2}{3} x^3 (3x^2 - 5) \\
\Rightarrow x = 0, \pm \sqrt[3]{\frac{5}{3}}
\]

**\( y \)-intercepts:**
\[
y = 6 \cdot 0^5 - 10 \cdot 0^3 = 0
\]

b) Is \( f(x) \) odd? even? neither?

\[
\begin{align*}
f(-x) &= (-x)^5 - 10 (-x)^3 \\
&= -\left(6x^5 - 10x^3\right) = -f(x)
\end{align*}
\]

Thus, odd function.

c) Find the absolute maximum and minimum of \( f(x) \) on the interval \([-2, 0]\).

\[
f'(x) = 30x^4 - 30x^2.
\]

\[
30x^4 - 30x^2 = 0 \Rightarrow 30x^2 (x^2 - 1) = 0
\]

\[
\begin{align*}
x &= 0, \pm 1 & \text{critical points.}
\end{align*}
\]

Only \(-1, 0 \in [-2, 0]\).

Compare:
\[
\begin{align*}
f(-2) &= 6(-32) - 10(-8) = -112 \\
f(-1) &= 6(-1) - 10(-1) = 4 \\
f(0) &= 0
\end{align*}
\]

absolute min

\[
\frac{\text{absolute max}}{\text{absolute min}}
\]

8
d) Find the interval(s) on which \( f(x) \) is increasing and the interval(s) on which it is decreasing.

\[
\begin{array}{c|c|c|c|c}
(\infty, -1) & (-1, 0) & (0, 1) & (1, \infty) \\
\hline
\text{f}' & + & - & - & + \\
\end{array}
\]

Hence, \( f \) is increasing on \( (-\infty, -1) \cup (1, \infty) \)

and \( f \) is decreasing on \( (-1, 0) \cup (0, 1) \)

e) Find all critical numbers for \( f(x) \) and classify any local maxima or minima.

From d), \( 0, \pm 1 \) are critical points of \( f \).

At \( x = -1 \), \( f \) has local maximum

At \( x = 0 \), neither max nor min

At \( x = 1 \), \( f \) has local minimum

f) Find the interval(s) on which \( f(x) \) is concave up and the interval(s) on which it is concave down.

\[
f'' = 120x^3 - 60x
\]

\[
f'' = 0 \implies 120x^3 - 60x = 0 \implies 60x(2x^2 - 1) = 0
\]

\[
\implies x = 0, \pm \sqrt{\frac{1}{2}}.
\]

\[
\begin{array}{c|c|c|c|c}
(\infty, -\sqrt{\frac{1}{2}}) & (-\sqrt{\frac{1}{2}}, 0) & (0, \sqrt{\frac{1}{2}}) & (\sqrt{\frac{1}{2}}, \infty) \\
\hline
f'' & - & + & - & + \\
\end{array}
\]

Down \quad up \quad up \quad down \quad up
g) Use the information from the previous parts of the problem and the fact that the graph of \( f(x) \) has inflection points at \((-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{4})\), \((0, 0)\), and \((\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{4})\) to sketch the graph of \( f(x) \). Label all inflection points, local maxima, and local minima.
7. (10 pts) Find the following limits. Write $\infty$ if it is infinity or DNE if it doesn’t exist.

a) \[ \lim_{x \to 0} \frac{x^2 + \sin(x)}{x} = \lim_{x \to 0} x + \lim_{x \to 0} \frac{\sin(x)}{x} = 0 + 1 = 1 \]

b) \[ \lim_{x \to -\infty} \frac{x^2}{e^x + 2} \]

\[ \lim_{x \to -\infty} \frac{\infty}{0+2} = \infty \]

c) \[ \lim_{x \to 0^+} x^x \]

Consider \[ \lim_{x \to 0^+} \ln x^x = \lim_{x \to 0^+} x \cdot \ln x \]

\[ = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \]

O'Hospital's Rule

\[ \lim_{x \to 0^+} x^x = e^{\lim_{x \to 0^+} \ln x^x} = e^{\lim_{x \to 0^+} \frac{1}{x} - x} = e^0 = 1 \]

\[ = 0 \]
8. (7 pts) Find the derivative of the function

\[ y = \frac{(x^2 + 3) \cos(x)}{e^x \ln(x^2 + 3)}. \]

**Logarithmic Differentiation:**

\[ \ln y = \ln (x^2 + 3) + \ln (\cos x) - \ln (e^x) - \ln (\ln(x^2 + 3)) \]

\[ \frac{d}{dx} \frac{dy}{dx} = \frac{2x}{x^2 + 3} - \frac{\sin x}{\cos x} - 1 - \frac{2x}{\ln(x^2 + 3)(x^2 + 3)} \]

\[ \frac{dy}{dx} = \left( \frac{2x}{x^2 + 3} - \frac{\sin x}{\cos x} - 1 - \frac{2x}{(x^2 + 3) \ln(x^2 + 3)} \right) \cdot \frac{(x^2 + 3) \cos x}{e^x \ln(x^2 + 3)} \]
9. (6 pts) Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 25 cm²?

\[ S(t) = l(t) \]

\[ \frac{dl}{dt} \]

When \( S(t) = 25 \text{ cm}^2 \), \( l(t) = 5 \text{ cm} \).

So \( \frac{dl}{dt} \) \( S(t) \) = \( 2 \cdot 5 \cdot 6 = 60 \text{ cm}^2/\text{s} \)

10. (7 pts) A farmer has 1200 ft of fencing and wants to fence off a rectangular field that borders a straight river. No fence is needed along the river. What are the dimensions of the field that has the largest area?

\[ \text{river} \]

\[ 1200 - 2x \]

\[ A(x) = x(1200 - 2x) \]

\[ A'(x) = (1200 - 2x) + x(-2) \]

\[ 1200 - 2x - 2x = 0 \]

\[ \Rightarrow x = 300. \]

At \( x = 300 \)

\[ A'' = -4 < 0 \] so local maximum

at \( x = 0 \), and \( x = 600 \) extremal situation

No area so

the dimension with largest area is \( 600 \times 300 \)
PART C.

11. (9 pts) Consider \( f(x) = x^2 \) on the interval \([0, 5]\).

a) Use a right-endpoint Riemann sum with 5 equal-length rectangles to approximate the area under the curve \( y = x^2 \) between \( x = 0 \) and \( x = 5 \).

\[
\int_{0}^{5} x^2 \, dx \approx 1.1 + 1.4 + 1.9 + 1.6 + 1.25 = 5.85.
\]

b) Is your answer in (a) larger or smaller than \( \int_{0}^{5} x^2 \, dx \)? Justify.

Larger. Because, \( x^2 \) is increasing.

Right endpoint gives overestimate.
12. (9 pts)

a) Express \( \lim_{n \to \infty} \sum_{i=1}^{n} \left[ (2i + 3) \left( \frac{4}{n} \right) \right] \) as a definite integral on the interval \([0, 4]\).

\( x_i \) should be \( \frac{4i}{n} \).

Then \( \int_{0}^{4} (2x+3) \, dx \n\)

b) Evaluate the integral found in part (a).

\[ \int_{0}^{4} (2x+3) \, dx = \left[ x^2 + 3x \right]_{0}^{4} \]

\[ = 28 \]
13. (16 pts) Suppose a ball is thrown upward from a 320 foot cliff with a velocity of 16 ft/sec.

a) Assuming that the air resistance can be ignored and that the acceleration due to gravity is $-32 \text{ ft/sec}^2$, how high does it go? That is, find the maximum height.

$$a = -32$$

$$V(t) = -32t + 16$$

$$h(t) = -16t^2 + 16t + 320.$$ 

Highest point $\iff V(t) = 0 \implies t = \frac{1}{2}.$

$$h\left(\frac{1}{2}\right) = -4 + 8 + 320 = 324 \text{ ft}.$$ 

b) At what time does the ball hit the ground?

Time hits the ground $\iff h(t) = 0$.

$$-16t^2 + 16t + 320 = 0$$

$\implies -t^2 + t + 20 = 0$

$$(5-t)(t+4) = 0 \implies t = 5 \text{ sec}.$$
14. (16 pts)

a) If \( F(x) = \int_{2}^{x} f(t) \, dt \) where \( f(z) = \int_{0}^{x^2+1} \sin(u^2) \, du \), find the second derivative of \( F \); that is, \( F''(x) \).

\[
F'(x) = f(x) = \int_{0}^{x^2+1} \sin(u^2) \, du
\]

\[
F''(x) = f'(x) = 2x \sin\left((x^2+1)^2\right) \cdot 2x
\]

b) Evaluate the integral

\[
\int_{0}^{4} \left[ \frac{d}{dt} \left( \frac{\ln(t^3+1)}{t^2+1} \right) \right] \, dt.
\]

Using the Fundamental Theorem of Calculus,

\[
= \left. \frac{\ln(t^3+1)}{t^2+1} \right|_{0}^{4}
\]

\[
= \frac{\ln(65)}{17} - \frac{\ln(1)}{1} = \frac{\ln(65)}{17}
\]
15. (16 pts) Evaluate the following integrals by interpreting each in terms of areas.

a) \( \int_{-5}^{0} \left( 2 + \sqrt{25 - x^2} \right) \, dx \)

\[ = \int_{-5}^{0} 2 \, dx + \int_{-5}^{0} \sqrt{25 - x^2} \, dx \]

\[ = 2 \cdot 5 + \frac{1}{4} \pi \left( 5 \right)^2 \]

\[ = 10 + \frac{25}{4} \pi \]

b) \( \int_{0}^{3} |x - 2| \, dx \)

\[ = 2 - 2 \cdot \frac{1}{2} + 1 \cdot 1 \cdot \frac{1}{2} \]

\[ = \frac{5}{2} \]
16. (20 pts) Evaluate the following integrals.

a) \[ \int_1^e \frac{1}{x} \, dx = \ln |x| \bigg|_1^e = 1 \]

b) \[ \int_0^\pi (\sin(x) + \cos(x)) \, dx \]

\[ = -\cos x + \sin x \bigg|_0^\pi = (1 + 0) - (-1 + 0) = 2 \]

c) \[ \int \frac{x^2 + 8}{x^2} \, dx = \int \frac{x^2}{x^2} \, dx + \int \frac{8}{x^2} \, dx \]

\[ = x + 8 \frac{x^{-1}}{-1} = x - 8x^{-1} + C \]

d) \[ \int (e^x + \frac{2}{1 + x^2}) \, dx \]

\[ = e^x + 2 \tan^{-1}(x) + C \]
17. (14 pts) A particle moves along a line so that its velocity a time $t$ is $v(t) = t^4 - 4t$ (measured in meters per second).

a) Find the displacement of the particle during the time interval $1 \leq t \leq 3$.

$$\text{displacement} = \int_1^3 t^4 - 4t \, dt$$

$$= \left[ \frac{t^5}{5} - \frac{4t^2}{2} \right]_1^3$$

$$= \left( \frac{243}{5} - 2.9 \right) - \left( \frac{1}{5} - 2 \right)$$

$$= \frac{242}{5} - 16$$

b) Find the distance traveled during the time period $1 \leq t \leq 3$.

$$\text{distance} = \int_1^3 |t^4 - 4t| \, dt$$

$$= \int_1^{3/4} (t^4 - 4t) \, dt + \int_{3/4}^3 (4t - t^4) \, dt$$

$t^4 - 4t = 0$, so $t = 0$ or $t = \sqrt[4]{4}$. Not good numbers. Change it to $v(t) = t^3 - 4t$.

So $t^3 - 4t = 0 \Rightarrow t = 0, \pm 2$.

$$= \int_1^3 |t^3 - 4t| \, dt$$

$$= \int_1^2 4t - t^3 \, dt + \int_2^3 t^3 - 4t \, dt$$

$$= 2t^2 \bigg|_1^2 - \frac{t^4}{4} \bigg|_1^2 + \frac{t^4}{4} - 2t^2 \bigg|_2^3$$

$$= 9$$