**Instructions:**

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT separate the pages of your exam.

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**Total** 100 ☐
Section A: Answer ALL questions.

Problem A1: [12 pts] In each case, find $\frac{dy}{dx}$.

(a) $y = \frac{xe^{-x}}{1+x^2}$

Solution:
Using the quotient-rule

$$\frac{dy}{dx} = \frac{(1+x^2) \frac{d}{dx}(xe^{-x}) - xe^{-x} \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)(e^{-x} - xe^{-x}) - xe^{-x}(2x)}{(1+x^2)^2}$$

$$= e^{-x} \frac{1-x-x^2-x^3}{(1+x^2)^2}$$

(b) $y = (\sin x)^x$

Solution:
Use logarithmic differentiation:

$$\ln y = x \ln(\sin x)$$

so

$$\frac{1}{y} \frac{dy}{dx} = \ln(\sin x) + x \frac{\cos x}{\sin x}$$

hence

$$\frac{dy}{dx} = (\sin x)^x \left( \ln(\sin x) + x \cot x \right).$$

(c) $xy^3 = 4$

Solution:
Use implicit differentiation:

$$y^3 + 3xy^2 \frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{y^3}{3xy^2} = -\frac{y}{3x}.$$
Problem A2: [16 pts] Find the following integrals.

(a) \( \int \frac{3}{1 + x^2} - \frac{1}{x^2} \, dx \)

Solution:

\[
\int \frac{3}{1 + x^2} - \frac{2}{x^3} \, dx = 3 \arctan(x) - 2 \left( \frac{1}{2} x^{-2} \right) + C = 3 \arctan(x) + \frac{1}{x^2} + C
\]

(b) \( \int \frac{\sin(\ln x)}{x} \, dx \)

Solution:

Let \( u = \ln x \) then \( du = \frac{1}{x} \, dx \) so

\[
\int \frac{\sin(\ln x)}{x} \, dx = \int \sin(u) \, du
= -\cos(u) + C
= -\cos(\ln x) + C
\]

(c) \( \int_1^2 x^3 - e^x \, dx \)

Solution:

\[
\int_0^2 x^3 - e^x \, dx = \left[ \frac{1}{4} x^4 - e^x \right]_1^2 = \left( \frac{16}{4} - e^2 \right) - (0 - 1) = 5 - e^2
\]

(d) \( \int_1^4 \frac{1}{1 + \sqrt{u}} \, dx \)

Solution:

Let \( u = \sqrt{x} \) then \( du = \frac{1}{2\sqrt{x}} \, dx \) so \( dx = 2u \, du \). When \( x = 1, u = \sqrt{1} = 1 \). When \( x = 4, u = \sqrt{4} = 2 \). Thus

\[
\int_{x=1}^4 \frac{1}{1 + \sqrt{u}} \, dx = \int_{u=1}^2 \frac{2u}{1 + u} \, du
= 2 \int_1^2 \frac{u + 1 - 1}{1 + u} \, du
= 2 \int_1^2 1 - \frac{1}{1 + u} \, du
= [u - \ln|1 + u|]_1^2 = (2 - \ln 3) - (1 - \ln 2)
= 1 - \ln \frac{3}{2}
\]
Problem A3: [4 pts] Two graphs of a function $f(x)$ are given below on axes with different scales.

(a) What are the critical values of $f(x)$?

Solution:

0, 1, 2

(b) Put the following $x$-values in order of increasing values of $f'(x)$. $x = -\frac{1}{2}, 0, \frac{3}{2}, 3$.

Solution:

$\frac{3}{2}, 0, -\frac{1}{2}, 3$

(c) At which of the following values is $f''(x)$ positive? $x = -1, \frac{1}{2}, 2, 3$.

Solution:

$\frac{1}{2}, 2$

(d) Is $\int_{\frac{1}{2}}^{3} f(x)dx$ positive or negative?

Solution:

Negative
Problem A4: [12 pts] Find the following limits:

(a) \( \lim_{x \to \infty} \frac{2x^3 - 3x^2 + 6}{(x + 2)^3} \)

Solution:
Divide top and bottom by \( x^3 \)

\[
\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 6}{(x + 2)^3} = \lim_{x \to \infty} \frac{2 - 3/x + 6/x^3}{(1 + 2/x)^3} = 2.
\]

(b) \( \lim_{x \to 0} x \sin x \)

Solution:
The form of the limit is \( 0/0 \) which is indeterminate, so we must take a logarithm and convert to a quotient so we can use L’Hôpital’s Rule.

\[
\lim_{x \to 0} \ln (x \sin x) = \lim_{x \to 0} x \sin x \ln x
\]

\[
= \lim_{x \to 0} \frac{\ln x}{\csc x}
\]

\[
L’H = \lim_{x \to 0} \frac{1/x}{- \csc x \cot x} = \lim_{x \to 0} - \frac{x \tan x}{x}
\]

\[
L''H = - \lim_{x \to 0} \frac{\cos x \tan x + x \sec^2 x}{1} = - 0 + 0 = 0
\]

This was however the \( \ln \) of the limit we needed to find, so

\[
\lim_{x \to 0} x \sin x = e^0 = 1.
\]
Problem A5: [10 pts] Ann is standing on a muddy field. Three kilometers due east is a road running north-south. Five kilometers down the road (north) is a farm. She can walk over the field at 2 km/h and along the road at 5 km/h. If she walks to the road and then along the road by the fastest route, where does she join the road?

Solution:
Set up a coordinate axis so that the road is the $y$-axis, Ann is standing at $(-3,0)$ and the farm is at $(0,5)$. Let the point at which she joins the road be $P = (0, y)$.

Her distance to $P$ is $\sqrt{9 + y^2}$. The distance from $P$ to the farm is $|5 - y|$.

It doesn’t make sense to walk south or further north than the farm, so we can take $0 \leq y \leq 5$. Thus we need to minimize

$$T(y) = \frac{1}{2} \sqrt{9 + y^2} + \frac{5 - y}{5} \quad \text{on} \quad 0 \leq y \leq 5.$$

Now $\frac{dT}{dy} = \frac{y}{2\sqrt{9 + y^2}} - \frac{1}{5}$ so $\frac{dT}{dy} = 0$ when $5y = 2\sqrt{9 + y^2}$. Squaring both sides yields $25y^2 = 18 + 2y^2$ so $y^2 = 6$. Since we only need to worry about $0 \leq y \leq 5$, the only important critical point is therefore $y = \sqrt{6}$.

Now $T(0) = \frac{3}{2} + 1 = \frac{5}{2}$, $T(\sqrt{6}) = \frac{\sqrt{15}}{2} + \frac{5 - \sqrt{15}}{5} \approx 2 + \frac{1}{5}$, $T(5) = \frac{\sqrt{34}}{2} \approx 3$

So $y = \sqrt{6}$ is indeed the minimum and she joins the road $\sqrt{6}$ km north of her current position.
Problem A6: [10 pts] A rocket is fired directly up from a resting position at ground level on a planet where the acceleration due to gravity is $4 \text{ m/s}^2$. For the first 5 seconds its acceleration is given by $a(t) = 6t \text{ m/s}$. After 5 seconds, its fuel supply runs out and it accelerates according to gravity. How high does it get?

Solution:

Let $s(t)$ represent its height. Then its acceleration is

$$s''(t) = \begin{cases} 6t, & t < 5 \\ -4, & t \geq 5 \end{cases}$$

Anti-differentiating yields

$$s'(t) = \begin{cases} 3t^2 + c_1, & t < 5 \\ -4t + c_2, & t \geq 5 \end{cases}$$

Now the rocket is fired from rest so $s'(0) = 0$ and $c_1 = 0$. By continuity of velocity, the values at $t = 10$ must match so $3(25)^2 = -4(5) + c_2$, so $c_2 = 75 + 20 = 95$. Thus

$$s'(t) = \begin{cases} t^2, & t < 5 \\ -4t + 95, & t \geq 5 \end{cases}$$

Then

$$s(t) = \begin{cases} t^3 + c_3, & t < 5 \\ -2t^2 + 95t + c_4, & t \geq 5 \end{cases}$$

Again $s(0) = 0$ so $c_3 = 0$. By continuity $125 = -50 + 475 + c_4$ so $c_4 = -300$ and

$$s(t) = \begin{cases} t^3, & t < 5 \\ -2t^2 + 95t - 300, & t \geq 5 \end{cases}$$

Now at the highest point $s'(t) = 0$ so we must solve $-4t + 95 = 0$ so $t = \frac{95}{4}$. The highest point is therefore

$$s\left(\frac{95}{4}\right) = -\frac{95^2}{8} + \frac{95^2}{4} - 300 = \frac{95^2}{8} - 300 = \frac{9025 - 2400}{8} = \frac{3625}{8}$$
Problem A7: [10 pts] A vertical wall 2 m high casts a shadow from the sun onto a flat field. At 6:00 pm, the shadow is 4 m long and growing at a rate of 3 m/hour. At what rate is the angle the sun makes with the ground changing at 6:00pm?

Solution:
Let \( x \) denote the length of the shadow and \( \theta \) the angle the sun makes with the ground. Then

\[
\tan \theta = \frac{2}{x}
\]

so using the chain-rule

\[
\sec^2 \theta \frac{d\theta}{dt} = \frac{-2}{x^2} \frac{dx}{dt}
\]

Now at 6:00pm, \( x = 4 \), \( \tan \theta = \frac{1}{2} \) so \( \sec^2 \theta = 1 + (1/2)^2 = \frac{5}{4} \) and \( \frac{dx}{dt} = 3 \). Thus

\[
\frac{5 d\theta}{4 dt} = \frac{-2}{4^2} \cdot 3
\]

and

\[
\frac{d\theta}{dt} = \frac{-6}{16} \cdot \frac{4}{5} = \frac{15}{2} \text{ rad/hour}.
\]
Problem A8: [10 pts] A car has velocity \( v(t) = \frac{3}{t+1} - 2 \)

(a) Find the net change in position over the interval \( 0 \leq t \leq 2. \)

Solution:

\[
s(2) - s(0) = \int_0^2 \frac{3}{t+1} - 2 \, dt = \left[ 3 \ln |t+1| - 2t \right]_0^2 = (3 \ln 3 - 6) - (3 \ln 1 - 0) = 3 \ln 3 - 6
\]

(b) Find the total distance traveled over the interval \( 0 \leq t \leq 2. \)

Solution:

\( v(t) = 0 \) when \( \frac{3}{t+1} = 2 \) so when \( 2t + 2 = 3, \) i.e. at \( t = \frac{1}{2}. \) And \( v(t) > 0 \) when \( t < 1/2, \) \( v(t) < 0 \) when \( t > 1/2. \)

Distance = \( \int_0^2 |v(t)| \, dt = \int_0^{1/2} \frac{3}{t+1} - 2 \, dt + \int_{1/2}^2 2 - \frac{3}{t+1} \, dt \)

\[
= \left[ 3 \ln |t+1| - 2t \right]_0^{1/2} + \left[ 2t - 3 \ln |t+1| \right]_{1/2}^2
= (3 \ln \frac{3}{2} - 1) - (0 - 0) + (6 - 3 \ln 3) - (1 - 3 \ln \frac{3}{2})
= 4 + 6 \ln \frac{3}{2} - 3 \ln 3
= 4 + 3 \ln 3 - 6 \ln 2
\]
Problem A9: [12 pts] Consider the function \( f(x) = x^3 - 6x^2 + 9x + 3 \)

(a) Find and classify all the critical points of \( f(x) \).

Solution:

\[
f'(x) = 3x^2 - 12x + 9 = 3(x - 3)(x - 1)
\]

so \( f'(x) = 0 \) when \( x = 1, 3 \). There are no points where the derivative is undefined, so \( x = 1, 3 \) are the only critical points.

\[
f''(x) = 6x - 12
\]

Then \( f''(1) = -6 < 0 \) so \( x = 1 \) is a local maximum and \( f''(3) = 6 > 0 \) so \( x = 3 \) is a local minimum.

(b) Find the absolute maximum and absolute minimum values that \( f(x) \) takes on the interval \([0, 2]\).

Solution:

We evaluate \( f(x) \) at the endpoints and at the critical points inside \([0, 2]\).

\[
f(0) = 3, \quad f(1) = 7, \quad f(2) = 5
\]

Thus the abs max value is 7, the abs min value is 3.

(c) On what interval(s) is \( f(x) \) concave up? Give your answer in interval notation.

Solution:

\( f(x) \) is concave up, where \( f''(x) > 0 \) so when \( 6x - 12 > 0 \). Therefore \( f(x) \) is concave up on \((2, \infty)\).
**Problem A10:** [4 pts] A differentiable function $f(x)$ satisfies

$$f(x + y) = f(x) + f(y) + f(x)f(y)$$

for all $x, y$. If $f(0) = 0$, $f(2) = 5$ and $f'(0) = 3$, find $f'(2)$.

**Solution:**

$$f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{5 + f(h) + 5f(h) - 5}{h}$$

But

$$3 = f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

so

$$f'(2) = 6 \cdot 3 = 18.$$ 

**Solution:**

(Method 2)

$$f(x + 2) = f(x) + 5 + 5f(x)$$

so the differentiating both sides yields

$$f'(x + 2) = f' + 5f'(x) = 6f'(x).$$

Putting in $x = 0$, we get

$$f'(2) = 6f'(0) = 18.$$