MTH161 Workshop 7: exponential growth and decay, related rates

Problem Set Instructions: Work through the following problems with your group. Pick a scribe who will write out your solutions on a separate sheet of paper and turn it in at the end of the session. You might not finish all of the problems, but be sure to work on all of them together and gain a good idea of how to proceed.

Discussion Questions: Discuss the following questions with your group.

• Discuss with your group the general strategy for solving related rates problems. What are you trying to do? What do you need in order to solve the problem?

1. Newton’s Law of Cooling says that the rate of change of the temperature of an object is proportional to the difference between the temperature $T$ of the object and the temperature $T_s$ of the surrounding environment. In other words, there is a constant $k$ such that
   $$\frac{dT}{dt} = k(T - T_s).$$
   (a) Verify that $T = T_s + Ce^{kt}$ is a solution to this differential equation, where $C$ is any constant.
   (b) A 100°C object is placed into a 25°C room. After ten minutes, the temperature of the object is 80°C. How long does it take for the object to reach a temperature of 60°C? What about 40°C? Will the object ever reach 25°C? Why or why not?
   (c) What happens to the temperature as $t \to \infty$?

2. Newton’s Law of Gravitation says that the magnitude $F$ of the force exerted by a body of mass $m$ on a body of mass $M$ is
   $$F = \frac{GmM}{r^2},$$
   where $G$ is the gravitational constant and $r$ is the distance between the bodies.
   (a) Find $\frac{dF}{dr}$ and explain its meaning. What does the minus sign indicate?
   (b) An object is traveling away from Earth. At the moment $r = 20,000$ km, the force that Earth is exerting on that object is decreasing at a rate of 2 N/km. (The Newton (N) is the unit of force, equivalent to a kilogram meter per second squared (kg·m/s²).) How fast was the force changing when $r$ was equal to 10,000 km?
3. A baseball diamond is a square with side length 90 ft. Jason hits the ball and runs from home plate toward first base at a speed of 26 ft/s. At the moment he is 30 ft from first base,
(a) at what rate is his distance from second base decreasing?
(b) at what rate is his distance from third base increasing?

4. The kinetic energy of a moving object is given by the formula

\[ K = \frac{1}{2}mv^2, \]

where \( m \) is the mass of the object and \( v \) is its velocity. The unit of energy is the Joule (J), or Newton-meter (N·m). Now, suppose a rock with mass \( m \) is falling toward the ground. At the moment that the rock’s speed is 30 m/s:
(a) What is the kinetic energy of the rock?
(b) How fast is the kinetic energy of the rock increasing?
(Hint: Remember that the acceleration due to gravity is 9.8 m/s\(^2\).)

5. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. With your group, determine how fast a rider is rising vertically when his seat is 16 m above ground level. (Remember, draw a picture first!)

Challenge Problem: A plane flying at a constant speed of 300 km/hr passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing one minute later?

(Hint: You’ll need the law of cosines, which says that if \( a, b, \) and \( c \) are the sides of a triangle, and \( \theta \) is the angle opposite the side of length \( c \), then \( c^2 = a^2 + b^2 - 2ab \cos \theta. \))