ON THE GLOBAL REGULARITY ISSUE OF THE
TWO-DIMENSIONAL MAGNETOHYDRODYNAMICS SYSTEM
WITH MAGNETIC DIFFUSION WEAKER THAN A LAPLACIAN

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ABSTRACT. In this manuscript, we discuss the recent developments in the re-
search direction concerning whether the solution to the two-dimensional mag-
netohydrodynamics system with certain velocity dissipation and magnetic dif-
fusion strengths preserves the high regularity of a given initial data for all time
or exhibits a blowup in finite time. In particular, we address an open prob-
lem in case the magnetic diffusion is weaker than a full Laplacian. In short,
we consider this system with both velocity dissipation and magnetic diffusion
strength measured in terms of fractional Laplacians with certain powers, and
point out the following gap in the results among the current literature. In
case the power of the fractional Laplacian representing the magnetic diffusion
is one, the global well-posedness follows as long as the velocity dissipation is
present regardless of how weak its strength may be, and hence the sum of the
two powers need to only be more than one (6, 31). On the other hand, once
the power of the fractional Laplacian representing the magnetic diffusion drops
below one, in order to ensure the system’s global well-posedness, the sum of
the powers from the dissipation and diffusion must be equal to or more than
two, improved only logarithmically (22, 23, 27). We discuss this issue in more
detail, explain its reasoning, and provide some basic regularity criteria to gain
better insight to this difficult direction of research.

Keywords: BMO space; Fourier transform; fractional Laplacian;
magnetohydrodynamics system; regularity.

1. Introduction

One of the most well-known outstanding open problems in mathematical anal-
ysis questions whether the solution to the following N-dimensional Navier-Stokes
equations (NSE), in case N = 3, admits a unique solution with finite kinetic energy
for all time or if there exists an initial data with finite kinetic energy such that its
corresponding solution experiences a blowup in finite time:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi - \nu \Delta u &= 0, \\
\nabla \cdot u &= 0, \quad u(x, 0) \equiv u_0(x),
\end{align*}
\]  

(1a) (1b)

where we denoted by u : \( \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N, \pi : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}, \) the velocity, the
pressure fields respectively, and \( \nu > 0 \) represents viscosity. This system at \( \nu = 0 \)
recovers the Euler equations.

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Hereafter let us write \( \frac{\partial}{\partial t} = \partial_t, \frac{\partial}{\partial x_i} = \partial_i \) for \( i \in \mathbb{N} \), \( x = (x_1, \ldots, x_N) \), as well as \( \int f = \int_{\mathbb{R}^N} f(x)dx \). Furthermore, for brevity we use the notations \( A \lesssim a, b \), \( A \approx a, b \) to imply that there exists a constant \( c(a, b) \) that depends on \( a, b \) such that \( A \leq cB, A = cB \) respectively. Finally, we denote the fractional Laplacian, \( \Lambda^r \triangleq (-\Delta)^{r/2}, r \in \mathbb{R} \), defined through a Fourier symbol of \( \xi \) so that

\[
\hat{\Lambda^r f}(\xi) = |\xi|^r \hat{f}(\xi).
\]

Let us now couple [1a] with Maxwell’s equation from electromagnetism to write down the generalized magnetohydrodynamics (MHD) system as

\[
\partial_t u + (u \cdot \nabla)u + \nabla \pi + \nu \Lambda^{2\alpha} u = (b \cdot \nabla)b, \quad (2a)
\]

\[
\partial_t b + (u \cdot \nabla)b + \eta \Lambda^{2\beta} b = (b \cdot \nabla)u, \quad (2b)
\]

\[
\nabla \cdot u = \nabla \cdot b = 0, \quad (u, b)(x, 0) \equiv (u_0, b_0)(x), \quad (2c)
\]

where we denoted by \( b : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N \), the magnetic field and \( \eta \geq 0 \) the magnetic diffusivity. Let us call the generalized MHD system at \( \nu, \beta > 0, \alpha = \beta = 1 \), the classical MHD system. As we will see, it is important to observe here that [2a] at \( \alpha = 1 \) is precisely [1a] with the forcing term of \( (b \cdot \nabla)b \). It is also worth noting that if one achieves in proving that an arbitrary initial data \((u_0, b_0)\) sufficiently smooth, its corresponding solution \((u, b)\) preserves its regularity for all time, then one may take \( b_0 \equiv 0 \), and deduce by uniqueness the smooth solution \( u \) for the NSE [1a]-[1b]. In this way, it may be argued that proving the global well-posedness of the MHD system [2a]-[2c] is more difficult than that of the NSE [1a]-[1b].

Now similarly to many other systems of equations in physics, the solutions to both the NSE [1a]-[1b] and the MHD system [2a]-[2c] admit a rescaling property, specifically that e.g. in the latter case, if \( (u, b)(x, t) \) and \( \pi(x, t) \) solves the generalized MHD system [2a]-[2c] for \( \gamma = \alpha = \beta \), then so does

\[
(u_\lambda, b_\lambda)(x, t) \triangleq \lambda^{2\gamma-1}(u, b)(\lambda x, \lambda^{2\gamma} t), \quad \lambda \in \mathbb{R}^+, \tag{3}
\]

(and \( \pi_\lambda(x, t) \triangleq \lambda^{4\gamma-2\pi}(x, \lambda^{2\gamma} t) \)). Moreover, taking \( L^2(\mathbb{R}^N) \)-inner products of \( (2a) - (2b) \) with \((u, b)\), making use of the divergence-free property of both \( u \) and \( b \) from [2c], the following identity that describes that conservation of energy and cumulative energy dissipation and diffusion may be shown:

\[
\left( \|u\|_{L^2}^2 + \|b\|_{L^2}^2 \right)(t) + \int_0^t (\|\Lambda^{2\alpha} u\|_{L^2}^2 + \eta \|\Lambda^{2\beta} b\|_{L^2}^2) \, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \tag{4}
\]

for all \( t \) in the time interval \([0, T]\) over which the solution exists. It is well-known that while other bounded quantities of the solutions exist (see [15]), those in [4] seem to be the most useful upon \textit{a priori} estimates in the effort to prove higher regularity of the solution. Studying the rescaled solution, e.g. \( u_\lambda(x, t) \), in this most useful bounded quantity deduces

\[
\|u_\lambda(x, t)\|_{L^2}^2 = \lambda^{4\gamma-2-N} \|u(\lambda^{2\gamma} t)\|_{L^2}^2 \tag{5}
\]

by (3), where \( 4\gamma - 2 - N = 0 \) if and only if \( \gamma = \frac{1}{2} + \frac{N}{4} \), indicating that \( \frac{1}{2} + \frac{N}{4} \) is a kind of a threshold for the powers \( \alpha, \beta \) such that relying only on these bounds from [4] in the \textit{a priori} estimates, one can hope to prove higher regularity only if both \( \alpha \) and \( \beta \) are equal to or more than \( \frac{1}{2} + \frac{N}{4} \). Indeed, this is precisely what was proven in [23]; the result therein at \( N = 2 \) in particular recovers the pioneering work of [14,16] that the NSE [1a]-[1b], and the classical MHD system respectively are both
Thus, multiplying (6) by $|w|^{p-2}w, p \geq 2$, integrating over $\mathbb{R}^2$ and using that
\[ \int (u \cdot \nabla)w|w|^{p-2}w = \frac{1}{p} \int (u \cdot \nabla)|w|^p = 0 \]
due to \((11b)\) lead to
\[
\frac{1}{p} \partial_t \|w\|^p_{L^p} = 0. \tag{8}
\]
Using that \(\frac{1}{p} \partial_t \|w\|^p_{L^p} = \|w\|_{L^p}^{-1} \partial_t \|w\|_{L^p}\), dividing \((8)\) by \(\|w\|_{L^p}^{-1}\), taking \(p \not\to +\infty\) shows that the sup\(_{t \in [0,T]} \|w(t)\|_{L^\infty}\) is bounded by \(\|w(0)\|_{L^\infty}\). With this bound attained, applying \(\Lambda^\alpha\) to \((1a)\) with \(\nu = 0, s > 2\), taking \(L^2(\mathbb{R}^2)\)-inner products with \(\Lambda^\alpha u\), employing \((36)\) from Lemma 3.1 and Lemma 3.2 lead immediately to
\[
\frac{1}{2} \partial_t \|\Lambda^\alpha u\|_{L^2}^2 = - \int [\Lambda^\alpha ((u \cdot \nabla)u) - (u \cdot \nabla)\Lambda^\alpha u] \cdot \Lambda^\alpha u \leq \|\nabla u\|_{L^\infty} \|\Lambda^\alpha u\|_{L^2}^2 \leq \left( \|u\|_{L^2} + \|w\|_{L^\infty} \log_2(1 + \|\Lambda^\alpha u\|_{L^2}^2) + 1 \right) \|\Lambda^\alpha u\|_{L^2}^2;
\]
hence, the higher regularity is attained via Gronwall’s inequality applied to \((9)\).
This recovers the classical results from \(35\) (see also \(10\)).

A natural question arises whether the favorable formulation of the vorticity equation may exploited in order to improve the results on the MHD system from \(16, 23\) that required \(\alpha \geq 1, \beta \geq 1\) in the 2d case. In this endeavor, applying a curl operator on \((2a)-(2b)\), one arrives at
\[
\begin{align*}
\partial_t w + (u \cdot \nabla)w + \nu \Lambda^{2\alpha} w &= (b \cdot \nabla)j, \\
\partial_t j + (u \cdot \nabla)j + \eta \Lambda^{2\beta} j &= (b \cdot \nabla)w + 2[\partial_t b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)] \tag{10b}
\end{align*}
\]
where \(j \triangleq \nabla \times b\) represents current density. A quick comparison of \((10a)\) with \((6)\) shows that \((10a)\) is still a transport equation, but now forced by \((b \cdot \nabla)j\) while dissipated fractionally by \(\nu \Lambda^{2\alpha} w\). Thus, upon an \(L^p(\mathbb{R}^2)\)-estimate of \(w\), in contrast to \((8)\), we are faced with
\[
\frac{1}{p} \partial_t \|w\|^p_{L^p} + \nu \int \Lambda^{2\alpha} w |w|^{p-2} w = \int (b \cdot \nabla) j |w|^{p-2} w. \tag{11}
\]
Therefore, we see heuristically that if \(\eta > 0\) and \(\beta\) is sufficiently large so that \((b \cdot \nabla)j\) is sufficiently smooth due to \((1)\) and thus bounded, then the higher regularity should be attained via an analogous computations in \((7)-(9)\). Moreover, even if \(\beta\) is not sufficiently large, if \(\nu > 0\) and \(\alpha > 0\) is adequately large, then \(|w|^{p-2} w\) within \(\int (b \cdot \nabla) j |w|^{p-2} w\) may be appropriately handled using \((4)\) and thus higher regularity may follow. Following this intuitive argument, an explosive amount of work appeared very recently improving results one after another.

On one hand, for the case \(\nu = 0\) and hence \(\alpha = 0\), Tran, Yu and Zhai in \(21\) showed that \(\beta > 2\) suffices in order to prove the global well-posedness. Jiu and Zhao in \(8\) and the author in \(26\) independently improved this result to \(\beta > \frac{3}{2}\). Thereafter, Jiu and Zhao in \(9\) and Cao, Wu and Yuan in \(24\) independently showed that \(\beta > 1\) in fact suffices; the approach of the former was taking advantage of the property of a heat kernel, while the latter the Besov space techniques (see also \(5\)). On the other hand, for the case \(\eta > 0, \beta = 1\), Tran, Yu and Zhai in \(21\) showed that \(\alpha \geq \frac{1}{2}\) suffices. Subsequently, Yuan and Bai in \(31\), as well as the author in \(28\), independently improved this result to \(\alpha > \frac{1}{3}\). Thereafter, Ye and Xu in \(33\) improved from \(\alpha > \frac{1}{3}\) to \(\alpha \geq \frac{1}{4}\). Finally, the authors in \(6\) proved that \(\alpha > 0\) suffices (also \(31\)). At the time of writing this current manuscript, the most difficult
remaining open problem is the case of \( \nu = 0 \) and hence \( \alpha = 0 \) while \( \eta > 0, \beta = 1 \), which would represent a complete extension of the classical results from [10] to the MHD system. We also refer to [20] for numerical analysis, and [7, 32] for regularity criteria results.

A common ingredient in the hypothesis of every one of these recent new results on the 2d generalized MHD system is that \( \eta > 0, \beta \geq 1 \). This is because applying a curl operator on (2a) not only gives a favorable vorticity formulation in the form of a forced dissipative transport equation in (10a), but furthermore if one combines an \( L^2(\mathbb{R}^2) \)-estimate of \( w \) with that of \( j \), even the forcing term \( (b \cdot \nabla) j \) vanishes due to a remarkable cancellation. Indeed, upon taking \( L^2(\mathbb{R}^2) \)-estimates of (10a)–(10b) with \( (w, j) \) respectively, summing them up leads to

\[
\frac{1}{2} \partial_t (\|w\|^2_{L^2} + \|j\|^2_{L^2}) + \nu \|\Lambda^\alpha w\|^2_{L^2} + \eta \|\Lambda^\beta j\|^2_{L^2} = \int 2(\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)) j
\]

where we made use of the facts that not only \( \eta > 0 \) implies that if \( \eta > 1 \) so that from (4) we would have \( \int_0^T \|\nabla b\|^2_{L^2} \, dt \lesssim 1 \), then Young’s inequality applied in (15) separating \( \frac{\eta}{2} \|\nabla j\|^2_{L^2} \) would allow us to close this \( (\|w\|^2_{L^2} + \|j\|^2_{L^2}) \)-estimate over \( [0, T] \). Let us point out, very importantly for our subsequent discussion, that from the first inequality of (15) we chose \( \|\nabla u\|_{L^2} \) to play the role of “being estimated” because \( \|\nabla u\|_{L^2} \approx \|w\|_{L^2} \), and we chose \( \|\nabla b\|_{L^4} \) and \( \|j\|_{L^4} \) to share the role of “to be absorbed to the diffusive term \( \eta \|\Lambda^\beta j\|^2_{L^2} \)” and “to be square-integrable in time.” This immediate \( H^1(\mathbb{R}^2) \)-bound of \( (u, b) \) in the 2d case is precisely the reason why the door was opened for all the recent results in [2, 6, 7, 8, 9, 21, 26, 28, 31, 32, 33, 34].

Now let us consider the same estimate in (15) with \( \beta < 1 \) and \( \alpha + \beta < 2 \) so that a global well-posedness in such a case will be an improvement of the work in [10]–[35]. One may follow the analogous idea in (15) and use Hölder’s inequality to bound the right-hand side of (12) by

\[
\int 2(\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)) j \lesssim \|\nabla b\|_{L^4} \|\nabla u\|_{L^4} \|j\|_{L^2}.
\]
However, we immediately realize that $\|\nabla b\|_{L^4}$ from (16) cannot possibly be interpolated between $\|\nabla b\|_{L^2}$ "being estimated," and $\|\Lambda^2 b\|_{L^2}$ which is "to be square-integrable in time," because $\beta < 1$. Therefore, this term must be interpolated as

$$\|\nabla b\|_{L^4} \lesssim \|\nabla b\|_{L^2}^{\frac{2\beta-1}{\beta}} \|\Lambda^2 \nabla b\|_{L^2}^{\frac{1}{\beta}},$$

(17)

i.e. partially "being estimated" and partially "to be absorbed to the diffusive term $\eta \|\Lambda^2 j\|_{L^2}^2.$ However, this application of Gagliardo-Nirenberg’s inequality in (17) requires $\beta \geq \frac{1}{2}$. On the other hand, since the interpolation of (17) did not make use of terms that are square-integrable in time from (4), it seems most efficient to interpolate $\|\nabla u\|_{L^4}$ from (16) between the “being estimated” and “to be square-integrable in time” as follows:

$$\|\nabla u\|_{L^4} \lesssim \|\nabla u\|_{L^2}^{\frac{2\alpha-3}{\alpha}} \|\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}.$$

(18)

However, this application of Gagliardo-Nirenberg’s inequality requires $\alpha \geq \frac{3}{2}$ and thus together with the requirement from (17), we already have the restriction of $\alpha + \beta \geq 2$. Various other options of Hölder’s inequality in (16) are possible; however, upon many attempts, every one of them seems to require $\alpha + \beta \geq 2$; in particular, the extreme case of

$$\int 2[\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)] j \lesssim \|\nabla b\|_{L^2}\|\nabla u\|_{L^\infty}\|j\|_{L^2}$$

immediately requires $\alpha > 2$ so that $\|\nabla u\|_{L^\infty} \lesssim \|u\|_{L^2} + \|\Lambda^\alpha u\|_{L^2}$ becomes integrable over $[0,T]$.

Heuristically the problem is that the right-hand side of (12) is more or less $|\nabla b|^2 |\nabla u|$; if this were $|\nabla u|^2 |\nabla b|$, then $H^1(\mathbb{R}^2)$-estimate of $(u,b)$ may be immediately attained with just $\nu > 0, \alpha = 1$ and $\eta = 0$. Unfortunately upon many attempts to make some cancellations to reverse such roles of $u$ and $b$, we found this task very difficult. Indeed, although in the literature there have been various identities due to unique cancellations via appropriate additions and subtractions, taking advantage of divergence-free conditions (e.g. [13, Lemma 2.3], [30, Proposition 1.1]), vorticity formulation is really remarkable in that upon applying a curl operator on the non-linear term,

$$\nabla \times ((u \cdot \nabla) u) = \partial_1 ((u \cdot \nabla) u_2) - \partial_2 ((u \cdot \nabla) u_1)$$

$$= \partial_1 u_1 \partial_1 u_2 + \partial_1 u_2 \partial_2 u_2 + (u \cdot \nabla) \partial_1 u_2$$

$$- \partial_2 u_1 \partial_1 u_1 - \partial_2 u_2 \partial_2 u_1 - (u \cdot \nabla) \partial_2 u_1 = (u \cdot \nabla) w,$$

cancellations occur within itself. One does not even need to add or subtract with another equation to make this happen!

One noteworthy idea, that has proven to be useful in the literature, is to consider a symmetric form of $S \triangleq w + j, \tilde{S} \triangleq w - j$ so that the roles of $u$ and $b$ are indeed somehow reversible. That is, we assume $\gamma = \alpha = \beta, \nu = \eta = \lambda$, estimate

$$\partial_1 S + ((u - b) \cdot \nabla) S + \lambda \Lambda^{2\gamma} S$$

$$= - \partial_1 u \cdot \nabla b_2 + \partial_2 u \cdot \nabla b_1 + \partial_1 b \cdot \nabla u_2 - \partial_2 b \cdot \nabla u_1,$$

(19a)

$$\partial_1 \tilde{S} + ((u + b) \cdot \nabla) \tilde{S} + \lambda \Lambda^{2\gamma} \tilde{S}$$

$$= \partial_1 u \cdot \nabla b_2 - \partial_2 u \cdot \nabla b_1 - \partial_1 b \cdot \nabla u_2 + \partial_2 b \cdot \nabla u_1,$$

(19b)
where we used that we may rewrite
\[ 2(\partial_t b_1(\partial_t u_2 + \partial_t u_1) - \partial_1 u_1(\partial_t b_2 + \partial_2 b_1)) = -\partial_t u \cdot \nabla b_2 + \partial_t u_2 \cdot \nabla b_1 + \partial_1 b \cdot \nabla u_2 - \partial_2 b \cdot \nabla u_1, \]  
(20)
and hope that we can complete the estimate of \( \|S\|_{L^2}^2 + \|\overline{S}\|^2_{L^2} \) with \( \gamma < 1 \) so that by triangle inequality we may conclude the bound of \( \|w\|_{L^2}^2 + \|j\|^2_{L^2} \). Unfortunately this strategy also seems to require \( \gamma \geq 1 \); e.g. assuming \( \gamma < 1 \), taking \( L^2(\mathbb{R}^2) \)-inner products with \( S \) in (19a), we may estimate
\[ \frac{1}{2} \partial_t \|S\|^2_{L^2} + \lambda \|\Lambda S\|_{L^2}^2 \lesssim (\|\partial_t u \cdot \nabla b_2\|_{L^r} + \|\partial_t u \cdot \nabla b_1\|_{L^r} + \|\partial_1 b \cdot \nabla u_2\|_{L^r} + \|\partial_2 b \cdot \nabla u_1\|) \|\Lambda S\|_{L^2} \]
\[ \leq \frac{\lambda}{4} \|\Lambda S\|^2_{L^2} + c\|\nabla u\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \]
\[ \leq \frac{\lambda}{4} \|\Lambda S\|^2_{L^2} + c\|\Lambda^{-\gamma} u\|_{L^2}^2 \|\nabla b\|_{L^2}^2 \]
by Hölder’s inequality, Hardy-Littlewood-Sobolev theorem (e.g. [18 pg. 119]), Young’s inequality and Sobolev embedding of \( \dot{H}^{1-\gamma}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \). For this estimate to be closed, in comparison with (4), we will need \( 2 - \gamma \leq \alpha \) which is if and only if \( 1 \leq \alpha \) and because we assumed \( \gamma = \alpha = \beta \), this requires \( \alpha + \beta \geq 2 \).
Therefore, we conclude that there has been this “large gap,” for a technical reason, that if \( \eta > 0, \beta \geq 1 \), the best results in the literature (e.g. [2, 5, 6, 9, 31]) requires only \( \alpha + \beta > 1 \). But once we reduce our hypothesis to \( \eta > 0, \beta < 1 \), then the requirement suddenly jumps to \( \alpha + \beta \geq 2 \) from the classical results ([16, 23]), improvable only logarithmically in [22, 24, 27]. In the rest of this paper we provide a basic Serrin-type regularity criterion for the solution to be smooth ([17]) to gain better insight to this difficult case when \( \eta > 0, \beta < 1 \) and \( \alpha + \beta < 2 \).

**Theorem 1.2.** Let \( N = 2, \nu > 0, \eta > 0, \beta \in [\frac{1}{2}, 1), \alpha \in [\frac{1}{2}, 2 - \beta) \) and
\[ (u, b) \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2(\mathbb{R}^2) \]
\[ \times C\left([0, T]; H^s(\mathbb{R}^2)\right) \cap L^2(\mathbb{R}^2) \]
(21)
be the solution pair to the generalized MHD system (2a)-(2c) for a given \( (u_0, b_0) \in H^s(\mathbb{R}^2), s > 2 \). Suppose the solution \((u, b)\) over the time interval of \([0, T]\) satisfies
\[ \int_0^T \|\nabla u\|_{L^p}^p \, d\tau < \infty \]
where \( \frac{2}{p} + \frac{2}{r} \leq 2 + \frac{2}{p} \left(1 - \frac{1}{\beta}\right), \]
(22)
where the case \( p = +\infty \) amounts to the condition of \( r \geq 1 \). Then \((u, b)\) remains in the same regularity class as (21) on \([0, T']\) for some \( T' > T \).

**Theorem 1.3.** Let \( N = 2, \nu > 0, \eta > 0, \beta \in [\frac{1}{2}, 1), \alpha \in [\frac{1}{2}, 2 - \beta) \) and \((u, b)\) in the regularity class of (27) be the solution pair to the generalized MHD system (2a)-(2c) for a given \((u_0, b_0) \in H^s(\mathbb{R}^2), s > 2 \). Suppose the solution \((u, b)\) over the
leads to second-order derivatives on \(b\) and \(j\) in this manuscript, we choose to focus on the hypothesis that \(\beta < 1\) and \(\partial_1 j\) will be difficult to close this estimate due to the second-order derivatives on \(b\). However, if we work on the \(H^s(\mathbb{R}^2)\)-space for \(s > 2\) instead of \(\nabla u\) for the MHD system starts with first integrating by parts and taking away the derivative of \(u\) in terms of \(L^2(\mathbb{R}^2)\)-norm of \(w\) and \(j\), one immediately realizes that integration by parts in the right side of (21) leads to second-order derivatives on \(b\) which we will not be able to estimate because \(\beta < 1\):

\[
\int [2\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)]j
= -2 \int u_2 \partial_1 ((\partial_1 b_1)j) + u_1 \partial_2 ((\partial_1 b_1)j) + 2 \int u_1 \partial_1 ((\partial_1 b_2 + \partial_2 b_1)j).
\]

Theorem 1.2 and Theorem 1.3 may not be too far out of reach.

Remark 1.1. Improvements on the space of initial data from \(H^s(\mathbb{R}^2)\) for \(s > 2\) is certainly possible. We choose not to pursue this direction of research here. We also emphasize that \(\lim_{\beta \to 1^-} 2 + \frac{1}{\beta}(1 - \frac{1}{\beta}) = 2\), indicating again that heuristically the well-posedness of the generalized MHD system (23) may not be too far out of reach.

Remark 1.2. A natural question is whether we may obtain a regularity criteria in terms of \(u\) or \(b\) instead of their gradients as in (22) and (23) respectively. We choose to leave this in the future direction of research, only commenting that naive attempts actually seem to fail immediately. This is because a standard procedure of obtaining such a regularity criteria e.g. in terms of \(u\) instead of \(\nabla u\) for the MHD system starts with first integrating by parts and taking away the derivative on \(u\). However, if we work on the \(H^1(\mathbb{R}^2)\)-estimate of \((u, b)\), then although by taking \(L^2(\mathbb{R}^2)\)-inner products on (2a)-(2b) with \((\Delta u, -\Delta b)\) respectively, one may compute

\[
\frac{1}{2} \partial_t (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \nu \|\Lambda^\alpha \nabla u\|_{L^2}^2 + \eta \|\Lambda^\beta \nabla b\|_{L^2}^2
\]

\[
= \int \langle u \cdot \nabla \rangle u \cdot \Delta u + \langle u \cdot \nabla \rangle b \cdot \Delta b - \int \langle b \cdot \nabla \rangle b \cdot \Delta u + \langle b \cdot \nabla \rangle u \cdot \Delta b
\]

\[
= \sum_{i,j,k=1}^{2} \int u_i \partial_i (\partial_j b_j \partial_k b_k) - \int u_j \partial_k (\partial_i b_i \partial_j b_j) + u_j \partial_i (\partial_i b_i \partial_j b_j),
\]

\[
\text{it will be difficult to close this estimate due to the second-order derivatives on } b \text{ and the hypothesis that } \beta < 1. \text{ Similarly in an attempt to estimate } L^2(\mathbb{R}^2) \text{-norm of } w \text{ and } j, \text{ one immediately realizes that integration by parts in the right side of (12) leads to second-order derivatives on } b \text{ which we will not be able to estimate because } \beta < 1.
\]

Proofs of Theorem 1.2 and Theorem 1.3

Local theory may be proven using mollifiers in a standard way (e.g. [15]); thus, in this manuscript, we choose to focus on \textit{a priori} estimates.
2.1. $H^1(\mathbb{R}^2)$-bound.

**Proposition 2.1.** Under the hypothesis of Theorem 1.2, the solution $(u, b)$ over the time interval $[0, T]$ satisfies

$$
\sup_{t \in [0, T]} (||u||^2_{L^2} + ||j||^2_{L^2})(t) + \int_0^T ||\Lambda^\alpha w||^2_{L^2} + ||\Lambda^\beta j||^2_{L^2} d\tau < \infty. \quad (24)
$$

**Proof.** Firstly, let us assume $p \in (\frac{2}{3}, \infty)$ in (22) and continue from (15), and estimate

$$
\int [2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)]j 
\lesssim ||\nabla u||_{L^p} ||\nabla b||_{L^2} ||\nabla b||_{L^\frac{2p}{p-2}} 
\lesssim ||\nabla u||_{L^p} ||\nabla b||_{L^2} ||\nabla b||_{L^\frac{2p}{p-2}} ||\Lambda^\beta \nabla b||_{L^\frac{2p}{p-2}} 
\lesssim \eta \|\Lambda^\beta j\|_{L^2}^2 + c||\nabla u||_{L^p} ||\nabla b||_{L^2}^2
$$

by Hölder’s inequality, Gagliardo-Nirenberg’s inequality and Young’s inequality.

Applying (25) in (15), subtracting $\frac{\eta}{2} ||\Lambda^\beta j||_{L^2}^2$ from both sides leads to

$$
\partial_t ||u||^2_{L^2} + ||j||^2_{L^2} + 2\nu ||\Lambda^\alpha w||^2_{L^2} + \eta \|\Lambda^\beta j\|_{L^2}^2 \lesssim ||\nabla u||_{L^p} \|\nabla b||_{L^2}^2 \quad (26)
$$

from which Gronwall’s inequality completes the proof of Proposition 2.1 in case $p \in (\frac{2}{3}, \infty)$. Next, the case $p = \frac{2}{3}$ is immediate as well because

$$
\int [2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)]j 
\lesssim ||\nabla u||_{L^\frac{2}{3}} ||\nabla b||_{L^2} ||\Lambda^\beta \nabla b||_{L^2} 
\lesssim \frac{\eta}{2} ||\Lambda^\beta \nabla b||_{L^2}^2 + c||\nabla u||_{L^\frac{2}{3}} ||\nabla b||_{L^2}^2
$$

by Hölder’s inequality, Sobolev embedding $\dot{H}^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{3}}(\mathbb{R}^2)$ and Young’s inequality. Finally, in the case $p = +\infty$ we immediately obtain

$$
\int [2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)]j \lesssim ||\nabla u||_{L^\infty} ||\nabla b||_{L^2}^2
$$

by Hölder’s inequality. After applying these estimates in (25), Gronwall’s inequality deduces (24) and completes the proof of Proposition 2.1. $\square$

**Proposition 2.2.** Under the hypothesis of Theorem 1.3, the solution $(u, b)$ over the time interval $[0, T]$ satisfies

$$
\sup_{t \in [0, T]} (||u||^2_{L^2} + ||j||^2_{L^2})(t) + \int_0^T ||\Lambda^\alpha w||^2_{L^2} + ||\Lambda^\beta j||^2_{L^2} d\tau < \infty. \quad (27)
$$
Proof. Firstly, let us consider the case $\frac{2}{p} + \frac{2}{r} \leq 2 + \frac{2}{p} \left(1 - \frac{1}{\alpha}\right)$, $\frac{2}{\beta} < p < \infty$. We continue again from (15) and estimate:

$$\int \left[2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)\right] j$$

$$\lesssim \|\nabla u\|_{L^2}\|\nabla b\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}$$

$$\lesssim \|\nabla u\|_{L^2}\|\nabla b\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}} \left\| \Lambda^\beta \nabla b \right\|_{L^2}$$

$$\lesssim \frac{\eta}{2} \left\| \Lambda^\beta \nabla b \right\|_{L^2}^2 + c \|\nabla u\|_{L^2}^2 \|\nabla b\|_{L^{\frac{2p}{p-2}}}^2$$

by Hölder’s inequality, Gagliardo-Nirenberg’s inequality and Young’s inequality. After applying (28) in (15), Gronwall’s inequality deduces (27) in this case. The case in which $p = \frac{2}{\beta}$ is immediately done similarly:

$$\int \left[2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)\right] j$$

$$\lesssim \|\nabla u\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^{\frac{2p}{p-2}}} \leq \frac{\eta}{2} \left\| \Lambda^\beta \nabla b \right\|_{L^2}^2 + c \|\nabla u\|_{L^2}^2 \|\nabla b\|_{L^{\frac{2p}{p-2}}}^2$$

by Hölder’s inequality, Sobolev embedding $\dot{H}^2(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{p-2}}(\mathbb{R}^2)$ and Young’s inequality.

Secondly, let us consider the case $\frac{2}{p} + \frac{2}{r} \leq 2 + \frac{2}{p} \left(1 - \frac{1}{\alpha}\right)$, $\max\{2, \frac{2}{\alpha}\} < p < \infty$. We estimate from (15):

$$\int \left[2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)\right] j$$

$$\lesssim \|\nabla u\|_{L^2}\|\nabla b\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}} \left\| \Lambda^\beta \nabla b \right\|_{L^2}$$

$$\lesssim \|\nabla u\|_{L^2}\|\nabla b\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}} \left(\|\Lambda^\alpha \nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right)$$

by Hölder’s inequality, Gagliardo-Nirenberg’s inequality and Young’s inequality. After applying (29) in (15), Gronwall’s inequality deduces (27).

Lastly, for the case $\int \|j\|_{BMO} d\tau < \infty$, we estimate:

$$\int \left[2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)\right] j$$

$$= \int \left[-\partial_1 u \cdot \nabla b_2 + \partial_2 u \cdot \nabla b_1 + \partial_1 b \cdot \nabla u_2 - \partial_2 b \cdot \nabla u_1\right] j$$

$$\lesssim \|\nabla u\|_{L^2}\|\nabla b\|_{L^2}\|\nabla b\|_{L^{\frac{2p}{p-2}}} \lesssim \|j\|_{BMO} \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right)$$

by (20), duality of BMO space and the Hardy space $H^1$, [4, Theorem 2.1] and Young’s inequality. After applying (30) to (15), Gronwall’s inequality deduces (27) and completes the proof of Proposition 2.2.

2.2. Higher regularity from $H^1(\mathbb{R}^2)$-bound. In this subsection, we raise the regularity from $H^1(\mathbb{R}^2)$-bound to $H^s(\mathbb{R}^2)$. As we emphasized in the Section 1, this is the relatively easier part of the proof; the difficulty in case $\beta < 1$ is the $H^1(\mathbb{R}^2)$-estimate which we already accomplished in Proposition 2.1 and Proposition 2.2.
Proposition 2.3. Under the hypothesis of Theorem 1.2 or Theorem 1.3, the solution \((u, b)\) over the time interval \([0, T]\) satisfies

\[
\sup_{t \in [0, T]} \left( \|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \right) + \int_0^T \left( \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Lambda^\beta \nabla j\|_{L^2}^2 \right) dt < \infty. \tag{31}
\]

\textbf{Proof.} Let us first assume the more difficult case \(\alpha < 1\). We take \(L^2(\mathbb{R}^2)\)-inner products on \((10a)-(10b)\) with \((-\Delta w, -\Delta j)\) to deduce

\[
\frac{1}{2} \partial_t (\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \nu \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \eta \|\Lambda^\beta \nabla j\|_{L^2}^2
\]

\[
= - \int \nabla u \cdot \nabla w \cdot \nabla w - \int \nabla u \cdot \nabla j \cdot \nabla j + 2 \int \nabla b \cdot \nabla j \cdot \nabla w + 2 \int \Lambda^{1-\beta} [\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)] \Lambda^{1+\beta} j
\]

after integration by parts. For clarify, let us estimate the first three integrals and the last integral separately. Firstly,

\[
- \int \nabla u \cdot \nabla w \cdot \nabla w - \int \nabla u \cdot \nabla j \cdot \nabla j + 2 \int \nabla b \cdot \nabla j \cdot \nabla w
\]

\[
\lesssim \|\nabla u\|_{L^{\frac{2}{1}}_t} \|\nabla w\|_{L^2} \|\nabla w\|_{L^{\frac{2}{1}}_t} + \|\nabla u\|_{L^{\frac{2}{1}}_t} \|\nabla j\|_{L^2} \|\nabla j\|_{L^{\frac{2}{1}}_t}
\]

\[
+ \|\nabla b\|_{L^{\frac{2}{1}}_t} \|\nabla j\|_{L^2} \|\nabla w\|_{L^{\frac{2}{1}}_t} + \|\nabla b\|_{L^{\frac{2}{1}}_t} \|\nabla j\|_{L^2} \|\nabla w\|_{L^{\frac{2}{1}}_t}
\]

\[
\lesssim \|\Lambda^{1-\alpha} \nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\Lambda^{1-\beta} \nabla j\|_{L^2} + \|\Lambda^{1-\alpha} \nabla u\|_{L^2} \|\nabla j\|_{L^2} \|\Lambda^{1-\beta} \nabla j\|_{L^2}
\]

\[
+ \|\Lambda^{1-\alpha} \nabla b\|_{L^2} \|\nabla j\|_{L^2} \|\Lambda^{1-\beta} j\|_{L^2}
\]

\[
\lesssim \nu \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \frac{\eta}{8} \|\Lambda^\beta \nabla j\|_{L^2}^2
\]

\[
+ c(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2)(\|\Lambda^{1-\alpha} \nabla u\|_{L^2}^2 + \|\Lambda^{1-\beta} j\|_{L^2}^2)
\]

by H"older’s inequalities, Sobolev embedding of \(\dot{H}^\gamma(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\gamma}}(\mathbb{R}^2), \dot{H}^{1-\gamma}(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{\gamma}}(\mathbb{R}^2)\), Lemma 3.3 and Young’s inequality. Secondly,

\[
2 \int \Lambda^{1-\beta} [\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)] \Lambda^{1+\beta} j
\]

\[
\lesssim \|\nabla b\|_{L^{\frac{2}{1-\beta}}_t} \|\nabla u\|_{L^2} \|\Lambda^{1-\beta} \nabla j\|_{L^2}
\]

\[
\lesssim \|\Lambda^{1-\beta} \nabla b\|_{L^{\frac{2}{1-\beta}}_t} \|\nabla u\|_{L^2} + \|\nabla b\|_{L^{\frac{2}{1-\beta}}_t} \|\Lambda^{1-\beta} \nabla u\|_{L^{\frac{2}{1-\beta}}_t}) \|\Lambda^{1-\beta} \nabla j\|_{L^2}
\]

\[
\lesssim \nu \|\Lambda^{1-\beta} j\|_{L^2}^2 + c(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2)(\|\Lambda^{1-\beta} w\|_{L^2}^2 + \|\Lambda^{1-\beta} j\|_{L^2}^2)
\]

by Hölder’s inequality, Lemma 3.4, Lemma 3.3, Sobolev embeddings of \(\dot{H}^{\beta}(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\beta}}(\mathbb{R}^2)\) and \(\dot{H}^{1-\beta}(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{\beta}}(\mathbb{R}^2)\)) and Young’s inequality. Applying \([33]\) and \([34]\) in \([32]\), using that \(\alpha, \beta \geq \frac{1}{2}\) by hypothesis, Gronwall’s inequality type argument using Proposition 2.1 and Proposition 2.2 deduces \([31]\).

The case in which \(\alpha \in \{1, 2, \beta\}\) is easier and it suffices for us to modify within the estimate of \([33]\) as follows:

\[
- \int \nabla u \cdot \nabla w \cdot \nabla w + 2 \int \nabla b \cdot \nabla j \cdot \nabla w
\]

\[
\lesssim \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\nabla w\|_{L^2} + \|\nabla b\|_{L^{\frac{2}{1-\beta}}_t} \|\nabla j\|_{L^2} \|\nabla w\|_{L^{\frac{2}{1-\beta}}_t}
\]

\[
\lesssim \nu \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \frac{\eta}{8} \|\Lambda^\beta \nabla j\|_{L^2}^2 + c(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2)(1 + \|\Lambda^\alpha w\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)
\]
Lemma 3.2. Along with the estimate of (33), and (34) to (32), we deduce (31) via Gronwall’s inequality. The proof of Proposition 2.3 is now complete.

Proofs of Theorem 1.2 and Theorem 1.3. The proofs of Theorem 1.2 and Theorem 1.3 follow immediately from Proposition 2.3, Lemma 3.2 and (37) from Lemma 3.1. Indeed, taking $L^2(\mathbb{R}^2)$-inner products of (2a)–(2h) with $(\Lambda^s u, \Lambda^s b)$, results in

$$\frac{1}{2} \partial_t (||\Lambda^s u||^2_{L^2} + ||\Lambda^s b||^2_{L^2}) + \nu ||\Lambda^{s+\alpha} u||^2_{L^2} + \eta ||\Lambda^{s+\beta} b||^2_{L^2}$$

which may be seen from (33), and (34) to (32), deduces (31) via Gronwall’s inequality. The proof of Proposition 2.3 is now complete. □

3. Appendix

For completeness, we state lemmas that have been used:

Lemma 3.1. (11; see also 26, Lemma 2.3 and Appendix] and 29, Lemma 2.7 and Appendix) Let $f \in H^s(\mathbb{R}^2)$, $s > 2$, satisfy $\nabla \cdot f = 0$, $\nabla \times f \in L^\infty(\mathbb{R}^2)$. Then

$$||\nabla f||_{L^\infty} \lesssim (||f||_{L^2} + ||\nabla \times f||_{L^\infty} \log_2 (2 + ||f||_{H^{s+1}}) + 1).$$

Similarly let $f \in L^2(\mathbb{R}^2) \cap W^{s, p}(\mathbb{R}^2)$ where $s \in \mathbb{R}$ such that $p \in [2, \infty)$, $\frac{s}{p} < s$. Then

$$||f||_{L^\infty} \lesssim (||f||_{L^2} + ||f||_{H^s} \log_2 (2 + ||f||_{W^{s, p}}) + 1).$$

Lemma 3.2. (12) Let $f, g$ be smooth such that $\nabla f \in L^{p_1}(\mathbb{R}^2)$, $\Lambda^{s-1} g \in L^{p_2}(\mathbb{R}^2)$, $\Lambda^s f \in L^{p_3}(\mathbb{R}^2)$, $g \in L^{p_4}(\mathbb{R}^2)$, where $p \in (1, \infty)$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $p_2, p_3 \in (1, \infty)$. Then

$$||\Lambda^s (fg) - f \Lambda^s g||_{L^p} \lesssim (||\nabla f||_{L^{p_1}} ||\Lambda^{s-1} g||_{L^{p_2}} + ||\Lambda^s f||_{L^{p_3}} ||g||_{L^{p_4}}).$$
Lemma 3.3. (e.g. Theorem 3.1.1 [3]) Suppose \( f \) satisfies \( \nabla \cdot f = 0, \nabla f \in L^p(\mathbb{R}^2), p \in (1, \infty) \). Then
\[
\|\nabla f\|_{L^p} \lesssim \frac{p^2}{p - 1} \|\nabla \times f\|_{L^p}.
\]

Lemma 3.4. (e.g. Lemma A.2 [11]) Let \( f \in W^{\delta,p_1}(\mathbb{R}^2) \cap L^{q_2}(\mathbb{R}^2) \) and \( g \in W^{\delta,p_2}(\mathbb{R}^2) \cap L^{q_1}(\mathbb{R}^2) \) where \( \delta \geq 0, 1 < p_k < \infty, 1 < q_k \leq \infty, \frac{1}{p_k} + \frac{1}{q_k} = \frac{1}{p} \) and \( k = 1, 2 \). Then
\[
\|fg\|_{W^{\delta,p}} \lesssim (\|f\|_{W^{\delta,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{q_2}} \|g\|_{W^{\delta,p_2}}).
\]

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References


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