GLOBAL WELL-POSEDNESS OF TRANSPORT EQUATION WITH NONLOCAL VELOCITY IN BESOV SPACES WITH CRITICAL AND SUPERCRITICAL DISSIPATION

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Abstract. We study the transport equation with nonlocal velocity introduced in [4]. We prove its global well-posedness under critical and supercritical dissipation, the last case under the smallness condition, in Besov spaces with critical and subcritical regularity indexes, using the Fourier localization method and modulus of continuity.

Keywords: Besov Space, Modulus of Continuity, Hilbert Transform

1. Introduction

We study a system of equations concerning the transport equation with nonlocal velocity in $\mathbb{R}$ defined as follows:

\begin{equation}
\partial_t \theta + u \theta_x + \nu \Lambda^{2\alpha} \theta = 0, \quad \theta(0, x) = \theta_0(x)
\end{equation}

where $\nu > 0$ is the dissipative coefficient which hereafter we shall assume to be one, with $H$ a Hilbert transform:

\[ u = H\theta \equiv \frac{1}{\pi} P.V. \int \frac{\theta(y)}{x-y} dy, \]

and the operator $\Lambda$ has its Fourier symbol $\hat{\Lambda} f = |\xi|^\alpha \hat{f}$ with $\alpha \in (0, \frac{1}{2}]$. Originally, the divergence-form of (1) was studied in [6] as a 1D model of Quasi-geostrophic equation (QG). The authors in [4] studied (1) as a simple model related to the vorticity form of 3D Euler equation, Constantin, Lax and Majda model and Birkhoff-Rott equations.

Hereafter let us denote (1) by CCF model for Córdoba-Córdoba-Fontelos model as known in the literature. We note that the original model consisted of a minus sign for the nonlinear term; however, to simplify our computation and due to the property of Hilbert transform, we decided to consider (1). We shall also refer to the case $\alpha > \frac{1}{2}$ the subcritical, $\alpha = \frac{1}{2}$ the critical and $\alpha < \frac{1}{2}$
the supercritical CCF model due to its rescaling \( \theta_{\lambda}(x,t) = \lambda^{2\alpha-1} \theta(\lambda x, \lambda^{2\alpha} t) \) and the fact that the sup norm of the solution to (1) is conserved (cf. [11]).

In [4], the authors showed that with symmetric positive initial data \( \theta_0(x) \) such that \( \max_x \theta_0(x) = 1 \) and \( \text{Supp}(\theta_0(x)) \subset [-L, L] \), the solutions to (1) with \( \nu = 0 \) experience \( \| \theta_x \|_{L^\infty} = \sup_x |\theta_x(\cdot, t)| \) blow up in finite time. On the other hand, with \( \alpha > 1/2 \), and \( \theta_0 \in H^2(R) \), the global regularity of the solution to (1) was shown as well as for \( \alpha = 1/2 \) with initial data sufficiently small.

Subsequently, the regularity results were improved in [21] and [11]. While the author in the former proved the global well-posedness of the critical CCF model with periodic smooth initial data, the author in the latter proved the global well-posedness of the critical and subcritical CCF model with an arbitrary initial data in \( H^{\max\{\frac{1}{2} - 2\alpha, 0\}} \) while the supercritical CCF model locally well-posed in \( H^{\frac{5}{2} - 2\alpha} \) and globally well-posed if the initial data is sufficiently small. Some polynomial-in-time decay estimates are also discussed in [11].

The blowup result was also improved. In [16], the authors showed that for \( \alpha \in [0, \frac{1}{4}), 0 < \delta < 1 - 4\alpha \) there exists a \( C = C(\alpha, \delta, s) \) such that if \( \int_0^\infty \theta_0(x) \frac{dx}{x^{1+\delta}} > C \)

where \( M = \| \theta_0 \|_{L^\infty} \), the solution to (1) blows up in finite time. In comparison, it is well-known that in the case of the Burgers equation which is essentially (1) with \( u = \theta \), the existence of initial data such that the solution blows up in finite time in the supercritical case and global well-posedness in the critical and subcritical cases are well-known (cf. [1], [14], [10]). We also note a recently introduced Burgers-Hilbert equation:

\[ \theta_t + \theta \theta_x = \Lambda^{2\alpha} H \theta \]

of which its blowup in the supercritical case was shown in [5]. The issue concerning the regularity and blowup of supercritical QG remains an important open problem (cf. [14]).

In this paper we obtain the local well-posedness of the CCF model in generalized Besov space, namely \( \dot{B}^s_{\dot{p},1} \) for all \( s \geq \frac{1}{p} + 1 - 2\alpha \) with \( p \) depending on \( \alpha \) and extend to globally in time for the case \( \alpha \geq \frac{1}{2} \) as well as \( \alpha \in (0, \frac{1}{2}) \) with small initial data, namely

**Theorem 1.1.** Let \( \theta_0 \in \dot{B}_{p,1}^s(R), s \geq \frac{1}{p} + 1 - 2\alpha \)

\[ p \in \begin{cases} [1, \infty) & \alpha = \frac{1}{2} \\ [\frac{1}{2\alpha}, \infty) & \alpha \in (0, \frac{1}{2}) \end{cases}. \]

Then there exists \( T > 0 \) such that the CCF model (1) has a unique solution

\[ \theta \in \tilde{L}_T^\infty \dot{B}_{p,1}^s \cap L_T^1 \dot{B}_{p,1}^{s+2\alpha} \]

Moreover, for the case \( \alpha = \frac{1}{2} \), for all \( \beta \in \mathbb{R}^+ \), \( t^\beta \theta \in \tilde{L}_T^\infty \dot{B}_{p,1}^{s+\beta} \).
**Theorem 1.2.** For $\alpha \geq \frac{1}{2}$, the CCF model (1) has a unique global solution

$$\theta \in C(\mathbb{R}^+; \dot{B}^s_{p,1}) \cap L^1_{loc}(\mathbb{R}^+; \dot{B}^{s+1}_{p,1}), \text{ where } s \geq \frac{1}{p} + 1 - 2\alpha$$

**Theorem 1.3.** For $\alpha \in (0, \frac{1}{2})$, $p \in \left[\frac{1}{2\alpha}, \infty\right]$, there exists a constant $\eta > 0$ such that if

$$\|\theta_0\|_{\dot{B}^s_{p,1}} < \eta, \quad s \geq \frac{1}{p} + 1 - 2\alpha$$

then the CCF model (1) admits a unique global solution

$$\theta \in C(\mathbb{R}^+; \dot{B}^s_{p,1}) \cap L^1_T(\mathbb{R}^+; \dot{B}^{s+2\alpha}_{p,1})$$

We note that another global well-posedness result under small initial data is obtainable generalizing the work in [23] on the supercritical QG. In this paper, we obtain a smallness condition similar to that in [13].

We note that $\dot{B}^s_{p,1}$ is critical Besov spaces for the CCF model; i.e.

$$\|\theta_\lambda(\cdot, t)\|_{\dot{B}^s_{p,1} + 1 - 2\alpha} \approx \|\theta(\cdot, \lambda^{2\alpha} t)\|_{\dot{B}^s_{p,1}}.$$

We state analogous results concerning global well-posedness of similar equations in critical Besov spaces: in [2] two-dimensional critical QG in $\dot{B}^0_{\infty,1}$, in [17] one-dimensional critical Burgers equation in $\dot{B}^\frac{1}{p}_{p,1}, p \in [1, \infty)$, in [19] N-dimensional Navier-Stokes equation particularly in $\dot{B}^\frac{N}{p} + 1 - 2\alpha_{2,\infty}$.

In contrast to the previous work in [21], the improvement concerning the periodicity is due to the spatial decay of the solution. In comparison to that of [11], our result is more general allowing all possible regularity index $s \geq \frac{1}{p} + 1 - 2\alpha$. In the critical case, in contrast to the case of [2] or [17], we obtain results allowing $\alpha \in (0, \frac{1}{2}]$ and in contrast to the case of [17] we will extend the local result to global in time under the smallness condition and consider the subcritical index as well. In the supercritical case, our result is similar to the work on [13] and [22] in which the authors constructed

$$\theta_0 \in \chi^s_p = \begin{cases} 
\dot{B}^s_{p,1} & \text{if } p < \infty \\
\dot{B}^s_{\infty,1} \cap \dot{B}^0_{\infty,1} & \text{otherwise}
\end{cases}$$

for $s \geq \frac{N}{p} + 1 - 2\alpha$ ($N = 2$ in the case [13]) and obtained global well-posedness of the supercritical QG and Porous Media equation under small initial data condition respectively. The reason for the restriction on $p$ in our case while theirs was $p \in [1, \infty)$ is the lack of divergence-free property. We mention that our result concerning the global well-posedness with the smallness condition applies for the Burgers equation with little modification.

We make a remark that it is not clear if a similar result in the space $\chi^s_p$ constructed above works for the CCF model due to the lack of $L^p$ maximum principle for the solutions to the CCF model for $p$ in general.
In the next section, we set notations and state preliminary results. We focus on the critical index \( s = \frac{1}{p} + 1 - 2\alpha \) and only mention key modification necessary to generalize the result for the subcritical in Appendix A.

2. Preliminary Notations and Results

We denote by \( \mathcal{S}(\mathbb{R}^N) \) the Schwartz class functions and \( \mathcal{S}'(\mathbb{R}^N) \), its dual, the space of tempered distributions. Define \( \mathcal{S}_0 \) to be the subspace of \( \mathcal{S} \) by

\[
\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}. \int_{\mathbb{R}^N} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \ldots \right\}
\]

Its dual \( \mathcal{S}'_0 \) is given by \( \mathcal{S}'_0 = \mathcal{S}/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P} \) where \( \mathcal{P} \) is the space of polynomials. For \( j \in \mathbb{Z} \) we define \( A_j = \{ \xi \in \mathbb{R}^N : 2^{j-1} < |\xi| < 2^{j+1} \} \). Then there exists a sequence \( (\Phi_j) \in \mathcal{S}(\mathbb{R}^N) \) such that

\[
supp \hat{\Phi}_j \subset A_j, \quad \hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jN}\Phi_0(2^j x) \quad \text{and}
\]

\[
\sum_{j=-\infty}^{\infty} \hat{\Phi}_j(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } \xi = 0 \end{cases}
\]

Consequently, for any \( f \in \mathcal{S}'_0 \),

\[
\sum_{j=-\infty}^{\infty} \Phi_j * f = f
\]

To define the homogeneous Besov space, we set

\[
\hat{\Delta}_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \ldots
\]

We define for \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \), the homogeneous Besov space \( \dot{B}^s_{p, r} \) by

\[
\dot{B}^s_{p, r} = \left\{ f \in \mathcal{S}'_0 : \| f \|_{\dot{B}^s_{p, r}} < \infty \right\}
\]

where

\[
\| f \|_{\dot{B}^s_{p, r}} = \begin{cases} \left( \sum_j (2^{js}\| \hat{\Delta}_j f \|_{L^p})^r \right)^{1/r} & \text{for } r < \infty \\ \sup_j 2^{js}\| \hat{\Delta}_j f \|_{L^p} & \text{for } r = \infty \end{cases}
\]

To define inhomogeneous Besov space, let \( \Psi \in C^\infty_0(\mathbb{R}^N) \) be even and satisfy

\[
1 = \hat{\Psi}(\xi) + \sum_{j=0}^{\infty} \hat{\Phi}_j(\xi)
\]

i.e. for any \( f \in \mathcal{S}' \),
\[ \Psi \ast f + \sum_{j=0}^{\infty} \Phi_j \ast f = f \]

We further set

\[ \triangle_j f = \begin{cases} 
0 & \text{if } j \leq -2 \\
\Psi \ast f, & \text{if } j = -1 \\
\Phi_j \ast f, & \text{if } j = 0, 1, 2, \ldots
\end{cases} \]

and define for any \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \), the inhomogeneous Besov space

\[ B^s_{p,r} = \{ f \in S' : \| f \|_{B^s_{p,r}} < \infty \} \]

where

\[ \| f \|_{B^s_{p,r}} = \left\{ \begin{array}{ll}
(\sum_{j=0}^{\infty} (2^j \| \triangle_j f \|_{L^p})^r)^{1/r}, & \text{if } r < \infty, \\
\sup_{-1 \leq j < \infty} 2^j \| \triangle_j f \|_{L^p}, & \text{if } r = \infty
\end{array} \right. \]

We use the usual notations of low-frequency cut-off \( \dot{S}_j \equiv \sum_{k < j} \dot{\triangle}_k \), and recall that \( \dot{\triangle}_j \dot{\triangle}_k \equiv 0 \) if \( |j - k| \geq 2 \) hence consequently

\[ \dot{\triangle}_j (S_{k-1} f \dot{\triangle}_k f) = 0 \quad \text{if } |j - k| \geq 4 \]

We now state some important results (cf. [7]):

**Lemma 2.1.** Bernstein’s Lemma: Let \( \alpha \geq 0, 1 \leq p \leq r \leq \infty \).

(a) If \( f \) satisfies \( \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^N : |\xi| \leq K2^j \} \) for some integer \( j \) and a constant \( K > 0 \), then

\[ \| \Lambda^{2\alpha} f \|_{L^r(\mathbb{R}^N)} \leq C_1 2^{2\alpha j + jN(\frac{1}{p} - \frac{1}{r})} \| f \|_{L^p(\mathbb{R}^N)} \]

(b) If \( f \) satisfies \( \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^N : K_1 2^j \leq |\xi| \leq K_2 2^j \} \) for some integer \( j \) and constants \( 0 < K_1 \leq K_2 \), then

\[ C_1 2^{2\alpha j} \| f \|_{L^r(\mathbb{R}^N)} \leq \| \Lambda^{2\alpha} f \|_{L^r(\mathbb{R}^N)} \leq C_2 2^{2\alpha j + jN(\frac{1}{p} - \frac{1}{r})} \| f \|_{L^p(\mathbb{R}^N)}, \]

where \( C_1 \) and \( C_2 \) are constants depending on \( \alpha, p, \) and \( r \) only.

**Proposition 2.2.** The following facts are also well-known:

(a). Generalized derivatives: Let \( \sigma \in \mathbb{R} \), then the operator \( \Lambda^\sigma \) is an isomorphism from \( \dot{B}^s_{p,r} \) to \( \dot{B}^{s-\sigma}_{p,r} \).

(b). Sobolev embedding: If \( p_1 \leq p_2 \) and \( r_1 \leq r_2 \), then \( \dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-N} \left( \frac{1}{p_1} - \frac{1}{r_2} \right)_{p_2} \)

(c). For \( (p, r) \in [1, \infty]^2 \) and \( s > 0 \), there exists a positive constant \( C = C(N, s) \) such that

\[ \| uv \|_{\dot{B}^s_{p,r}} \leq C(\| u \|_{L^\infty} \| v \|_{\dot{B}^s_{p,r}} + \| v \|_{L^\infty} \| u \|_{\dot{B}^s_{p,r}}) \]
We use coupled space-time Besov spaces. Firstly, for $T > 0$, $\rho \in [1, \infty]$, 

\[ L^p([0, T], \dot{B}_{p,r}^s) = L^p_T \dot{B}_{p,r}^s = \{ f \in S'_0 : \| f \|_{L^p_T \dot{B}_{p,r}^s} = \| (\sum_{q \in \mathbb{Z}} 2^{qs} \| \triangle_q f \|_{L^p})^{\frac{1}{2}} \|_{L^p_T} < \infty \} \]

and the second mixed space is 

\[ \tilde{L}^p([0, T], \dot{B}_{p,r}^s) = \tilde{L}^p_T \dot{B}_{p,r}^s = \{ f \in S'_0 : \| f \|_{\tilde{L}^p_T \dot{B}_{p,r}^s} = \| (\sum_{q \in \mathbb{Z}} 2^{qs} \| \dot{\triangle}_q f \|_{L^p_{T,r}}) \|_{L^p_T} < \infty \} \]

Due to Minkowski's inequality, we have 

\[ \| f \|_{L^p_T \dot{B}_{p,r}^s} \leq \| f \|_{\tilde{L}^p_T \dot{B}_{p,r}^s} \quad \text{if } \rho \leq r \]
\[ \| f \|_{\tilde{L}^p_T \dot{B}_{p,r}^s} \geq \| f \|_{L^p_T \dot{B}_{p,r}^s} \quad \text{if } \rho \geq r \]

The following is due to [13]:

Lemma 2.3. Let $\phi$ be a smooth function supported in the shell $\{ x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2, 0 < R_1 < R_2 \}$. There exist two positive constants $\kappa$ and $C$ depending only on $\phi$ such that for all $1 \leq p \leq \infty$, $\tau \geq 0$ and $\lambda > 0$, we have 

\[ \| \phi(\lambda^{-1}|D|) e^{-\tau \lambda^2 \alpha} u \|_{L^p} \leq C e^{-\kappa \tau \lambda^2 \alpha} \| \phi(\lambda^{-1}|D|) u \|_{L^p} \]

Next result is due to [12]:

Lemma 2.4. Let $v$ be a smooth vector field. Let $\psi_t$ be a solution to 

\[ \psi_t(x) = x + \int_0^t v(\tau, \psi(\tau)) d\tau \]

Then for all $t \in \mathbb{R}$, the flow $\psi_t$ is a $C^1$ diffeomorphism over $\mathbb{R}^N$ and 

\[ \| \nabla \psi_t^{\pm 1} \|_{L^\infty} \leq e^{V(t)} \]
\[ \| \nabla \psi_t^{\pm 1} - Id \|_{L^\infty} \leq e^{V(t)} - 1 \]
\[ \| \nabla^2 \psi_t^{\pm 1} \|_{L^\infty} \leq e^{V(t)} \int_0^t \| \nabla^2 v(\tau) \|_{L^\infty} e^{V(\tau)} d\tau, \]

where $V(t) = \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau$.

The following is due to [8]:

Lemma 2.5. Let $v$ be a given vector field in $L^1_{loc}(\mathbb{R}^+; \text{Lip}(\mathbb{R}^N))$. For $q \in \mathbb{Z}$, we set $u_q = \triangle_q u$ and denote by $\psi_q$ the flow of the regularized vector field $\hat{S}_{q-1} v$. Then for $u \in B^0_{p,\infty}$ with $\alpha \in [0, 1)$ and $p \in [1, \infty]$ for some $C = C(\alpha, p) > 0$, 

\[ \| \Lambda^{2\alpha}(u_q \circ \psi_q) - (\Lambda^{2\alpha} u_q) \circ \psi_q \|_{L^p} \leq C e^{CV(t)} V^{1-\alpha}(t) 2^{q2\alpha} \| u_q \|_{L^p} \]
where $V(t)$ is same as that in the previous Lemma.

The following two results from [17] and [13] concern the Transport-Diffusion equation defined as

$$(TD)_{\nu,\alpha} \quad \partial_t \theta + v \cdot \nabla \theta + \nu \Lambda^{2\alpha} \theta = f, \quad \theta(x,0) = \theta_0(x)$$

where $v$ does not need to be divergence-free:

**Lemma 2.6.** Let $\sigma \in \mathbb{R}$ and $1 \leq p \leq p_1 \leq \infty$. Let $R_q = (\dot{S}_{q-1} v - v) \cdot \nabla \dot{\Delta}_q \theta - [\dot{\Delta}_q, v \cdot \nabla] \theta$. There exists a constant $C = C(N, \sigma)$ such that

$$2^{q'} \| R_q \|_{L^p} \leq C \left[ \sum_{|q'-q| \leq 4} \| \dot{S}_{q'-1} \nabla v \|_{L^\infty} 2^{q' \sigma} \| \dot{\Delta}_{q'} \theta \|_{L^p} \right] + \sum_{q' \geq q-3} 2^{q'-q} \| \dot{\Delta}_{q'} \nabla v \|_{L^\infty} 2^{q' \sigma} \| \dot{\Delta}_{q'} \theta \|_{L^p}$$

$$+ \sum_{|q'-q| \leq 4, q' \leq q-2} 2^{(q-q')(\sigma-1)} \left( \frac{N}{|q'|} \right)^{2/q'} \left( \frac{N}{|q'|} \right)^{2/q'} \| \dot{\Delta}_{q'} \nabla v \|_{L^p} 2^{q' \sigma} \| \dot{\Delta}_{q'} \theta \|_{L^p}$$

$$+ \sum_{q' > q-3, |q'-q| \leq 1} 2^{(q-q')(\sigma+N\min(\frac{1}{p_1}, \frac{1}{r}))} \left( \frac{N}{|q'|} \right)^{2/q'} \| \dot{\Delta}_{q'} \nabla v \|_{L^p} 2^{q' \sigma} \| \dot{\Delta}_{q'} \theta \|_{L^p}$$

**Lemma 2.7.** Let $1 \leq p_1 \leq p \leq \infty, 1 \leq r \leq p_1 \leq \infty$ and $1 \leq r \leq \infty$. Let $s \in \mathbb{R}$ satisfy

$$s < 1 + \frac{N}{p_1} \text{ (or } s \leq 1 + \frac{N}{p_1} \text{ if } r = 1),$$

$$s > -N \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \text{ (or } s > -1 - N \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \text{ if div}(v) = 0).$$

There exists a constant $C > 0$ depending only on $N, \alpha, s, p, p_1$ and $r$ such that for any smooth solution $\theta$ of $(TD)_{\nu,\alpha}$, we have the following a priori estimate:

$$\nu \frac{1}{2} \| \theta \|_{L^p_{T} B^s_{p,r}} + \nu^{\frac{1}{2}} \| f \|_{L^p_{T} B^s_{p,r} - 2\alpha + \frac{2\alpha}{p}} \leq C e^{CZ(T)} \left( \| \theta_0 \|_{B^s_{p,r}} + \nu^{\frac{1}{2}} \| f \|_{L^p_{T} B^s_{p,r} - 2\alpha + \frac{2\alpha}{p}} \right)$$

where $Z(T) = \int_0^T \| \nabla v(t) \|_{B^{0}_{p,1}\cap L^\infty} dt$. Besides, if $v = H \theta$, we have above estimate valid for all $s > 0$ with $Z(T) = \int_0^T \| \partial_x v \|_{L^\infty} dt$.

Finally, the following is a Lemma A.1 in [12]:

**Lemma 2.8.** Let $\chi \in \mathcal{S}(\mathbb{R}^N)$. There exists a constant $C = C(\chi, N)$ such that for all $C^2$ diffeomorphisms $\psi$ over $\mathbb{R}^N$ and $\psi^{-1} = \phi$, for all $\theta \in \mathcal{S}'(\mathbb{R}^N)$, for all $p \in [1, +\infty]$ and $(q, q') \in \mathbb{Z}^2$, we have

$$\| \chi(2^{-q'}D)(\dot{\Delta}_q \theta \circ \psi) \|_{L^p} \leq \| J_{\phi} \|_{L^\infty} \frac{1}{2^{-q'}} \| \dot{\Delta}_q \theta \|_{L^p} 2^{-q'} \| \nabla J_{\phi} \|_{L^\infty} \| J_{\psi} \|_{L^\infty} + 2^{q'-q'} \| \nabla \phi \|_{L^\infty}$$
where $J_\phi$ is the Jacobian determinant of $\phi$.

3. Proof of Local Result

In this section we prove Theorem 1.1 in steps of contraction argument:

3.1. Step 1: Approximating Solution. Let $\theta^0(x, t) = e^{-t\Lambda^{2\alpha}} \theta_0(x)$, $u^0 = H \theta_0$ and $\theta^{n+1}$ solve the system of linear equations:

\[
\begin{align*}
\partial_t \theta^0 + \Lambda^{2\alpha} \theta^0 &= 0 \\
\theta^0(0, x) &= \theta_0(x)
\end{align*}
\]

\[
\begin{align*}
\partial_t \theta^{n+1} + u^n \partial_x \theta^{n+1} + \Lambda^{2\alpha} \theta^{n+1} &= 0 \\
\theta^{n+1}(0, x) &= \theta_0(x), u^n = H \theta^n
\end{align*}
\]

for $n = 0, 1, 2, \ldots$ We have

\[
\|\theta^0\|_{L^1(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1} \cap \dot{B}^{1-2\alpha}_{p,1})} \lesssim \int_{\mathbb{R}^+} \sum_{j \in \mathbb{Z}} 2^{j \frac{1}{p} (1-2\alpha)} 2^{j/2} e^{-\tau 2^{j/2} \alpha} \|\theta_0\|_{L^p} d\tau \lesssim \|\theta_0\|_{\dot{B}^{1-2\alpha}_{p,1}}
\]

by Lemma 2.3. Thus, $\theta^0(t, x) \in L^1(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1})$. By definition of $\theta^0$ and the fact that $\theta_0 \in \dot{B}^{1-2\alpha}_{p,1}$, we have $\theta^0 \in \dot{L}^\infty(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1})$ and thus,

\[
\theta^0 = e^{-t\Lambda^{2\alpha}} \theta_0 \in \dot{L}^\infty(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1}) \cap L^1(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1})
\]

Suppose this holds for $n$; we will prove it true for $n + 1$ using Lemma 2.7. For Lemma 2.7 we take $p = \infty$, $r = 1$, $\rho = \infty$ and $s = \frac{1}{p} + 1 - 2\alpha$ so require $0 < \frac{1}{p} + 1 - 2\alpha \leq 1$. Hence,

\[
p \in \begin{cases} \{1, \infty\}, & \alpha = 1/2 \\
\{\frac{1}{2\alpha}, \infty\}, & \alpha \in (0, \frac{1}{2}) \end{cases}
\]

With such choice, by Lemma 2.7 we have

\[
\|\theta^{n+1}\|_{L^1(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1} \cap \dot{B}^{1-2\alpha}_{p,1})} \leq C e^T \|\theta^0\|_{\dot{L}^\infty(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1})}.
\]

Moreover, $\dot{B}^{1-2\alpha}_{p,1} \hookrightarrow \dot{B}^{0}_{\infty,1} \hookrightarrow \dot{L}^\infty$ by Proposition 2.2(b). Therefore,

\[
\|\theta^{n+1}\|_{L^1(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1} \cap \dot{B}^{1-2\alpha}_{p,1})} \leq C e^T \|\theta^0\|_{\dot{L}^\infty(\mathbb{R}^+, \dot{B}^{1-2\alpha}_{p,1})} < \infty
\]

by the induction hypothesis and continuity of Hilbert transform in homogeneous Besov spaces. On the other hand,
\[ \| \theta_{n+1} \|_{L^p_t L^\frac{1}{p}_x} \leq c \| \theta_{n+1} \|_{L^1_t L^\frac{1}{p}_x} \leq c e^{\int_0^T \| \partial_x u^n \|_{B^{\alpha}_{p,1}} L^\alpha dt} \| \theta_0 \|_{B^{\frac{1}{p}+1-2\alpha}_{p,1}} < \infty \]

by Lemma 2.7 with \( \rho = 1, r = 1, s = \frac{1}{p} + 1 - 2\alpha, p_1 = \infty. \) This implies for all \( n \in \mathbb{N}, \theta^n \in \dot{L}^{\infty}(\mathbb{R}^+, B^{\frac{1}{p}+1-2\alpha}_{p,1}) \cap L^1(\mathbb{R}^+, B^{\frac{1}{p}+1}_{p,1}). \]

3.2. Step 2: Uniform Bounds. Here we obtain uniform bounds with respect to parameter \( n \) for some \( T > 0 \) independent of \( n. \)

Let \( \theta_q = \Delta_q \theta \) and applying \( \Delta_q \) to (2) we obtain

\[ \partial_t \theta_{q+1} + (\dot{S}_{q-1} u^n)(\theta_{q+1})_x + \Lambda^{2\alpha} \theta_{q+1} = R_q \]

\[ := (\dot{S}_{q-1} u^n - u^n)(\theta_{q+1})_x - [\Delta_q, u^n \partial_x] \theta_{q+1} \]

\[ = (\dot{S}_{q-1} u^n - u^n)(\theta_{q+1})_x - \Delta_q u^n \partial_x \theta_{q+1} + u^n \partial_x \Delta_q \theta_{q+1} \]

Let \( \psi_q \) be the flow of the regularized \( \dot{S}_{q-1} u^n \) and denote \( \tilde{\theta}_q = \theta_q \circ \psi_q, \tilde{R}_q = R_q \circ \psi_q \) and \( \tilde{G}_q = \Lambda^{2\alpha} (\theta_{q+1} \circ \psi_q) - (\Lambda^{2\alpha} \theta_{q+1}^n) \circ \psi_q. \) Then we have

\[ \partial_t \tilde{R}_q + \Lambda^{2\alpha} \tilde{R}_{q+1} = \tilde{R}_q + \Lambda^{2\alpha} (\theta_{q+1} \circ \psi_q) - (\Lambda^{2\alpha} \theta_{q+1}^n) \circ \psi_q = \tilde{R}_q + \tilde{G}_q \]

Applying \( \Delta_j \) to this we get

\[ \Delta_j \tilde{R}_{q+1}^n(t) = e^{-\Lambda^{2\alpha} t} \Delta_j \theta_{0,q} + \int_0^t e^{-(t-\tau)\Lambda^{2\alpha}} (\Delta_j \tilde{R}_q + \Delta_j \tilde{G}_q) d\tau \]

which gives

\[ \| \Delta_j \tilde{R}_{q+1}^n(t) \|_{L^p} \lesssim \| e^{-\Lambda^{2\alpha} t} \Delta_j \theta_{0,q} \|_{L^p} + \int_0^t \| e^{-(t-\tau)\Lambda^{2\alpha}} (\Delta_j \tilde{R}_q + \Delta_j \tilde{G}_q) \|_{L^p} d\tau \]

By Lemma 2.3 we have

\[ \| \Delta_j \tilde{R}_{q+1}^n(t) \|_{L^p} \lesssim e^{-\kappa t^{2\alpha}} \| \Delta_j \theta_{0,q} \|_{L^p} + \int_0^t e^{-\kappa(t-\tau)^2\alpha} [\| \Delta_j \tilde{R}_q \|_{L^p} + \| \Delta_j \tilde{G}_q \|_{L^p}] d\tau \]

for some \( \kappa > 0. \) Now by Lemma 2.5 we have

\[ \| \Delta_j \tilde{G}_q(t) \|_{L^p} \leq C e^{CV(t)} V^{1-\alpha}(t) 2^{2\alpha} \| \theta_{q+1} \|_{L^p} \]

where \( V(t) = \int_0^t \| \partial_x u^n(\tau) \|_{L^\infty} d\tau. \) On the other hand,
(5) \[ \| \hat{\Delta}^j \bar{R}_q(t) \|_{L^p} \lesssim 2^{-j \| \partial_x \hat{\Delta}^j \bar{R}_q(t) \|_{L^p} \]
\[ \lesssim 2^{-j \| (\partial_x R_q) \circ \psi_q \|_{L^p} \| \partial_x \psi_q \|_{L^\infty}} \lesssim 2^{-j \| \partial_x R_q \|_{L^p} \| J_{\psi_q} \|_{L^\infty}^{\frac{1}{2}} \| \partial_x \psi_q \|_{L^\infty}} \lesssim 2^{-j \| R_q \|_{L^p} \| J_{\psi_q}^{-1} \|_{L^\infty}^{\frac{1}{2}} \| \partial_x \psi_q \|_{L^\infty}} \lesssim e^{CV(t)} 2^{-j \| R_q \|_{L^p}} \]
by Bernstein’s inequality and Lemma 2.4. This gives

(6) \[ \| \hat{\Delta}^j \bar{q}^{n+1}(t) \|_{L^p} \lesssim e^{-\kappa t 2^{2q_n}} \| \hat{\Delta}^j \theta_{0,q} \|_{L^p} \]
\[ + \int_0^t e^{-\kappa(t-\tau)2^{2q_n}} e^{CV(\tau)} \| V^{1-\alpha}(\tau) 2^{q_n} \|_{L^p} + 2^{q_n} \| \theta_{0,q} \|_{L^p} \]d\tau

We take \( L^2 \) norm over \([0, t]\) and multiply by \( 2^{(q+\alpha)} \) to obtain

(7) \[ 2^{(q+\alpha)} \| \hat{\Delta}^j \bar{q}^{n+1}(t) \|_{L^2_{t,L^p}} \lesssim 2^{(q-j)\alpha} (1 - e^{-\kappa t 2^{2q_n}}) \| \hat{\Delta}^j \theta_{0,q} \|_{L^p} \]
\[ + 2^{(q+\alpha)} \| \theta_{0,q} \|_{L^p} \]
\[ + 2^{(q-j)(1+\alpha) + \alpha} \int_0^t e^{CV(\tau)} \| R_q \|_{L^p} d\tau \]

Fix \( M_0 \in \mathbb{Z} \) to be specified later. Decompose

\[ \theta_{q}^{n+1} = \hat{S}_{q-M_0} \bar{q}^{n+1} \circ \psi_q^{-1} + \sum_{j \geq q-M_0} \hat{\Delta}^j \bar{q}^{n+1} \circ \psi_q^{-1} \]

Then, for all \( t \in [0, T] \),

(8) \[ \| \theta_{q}^{n+1} \|_{L^2_{t,L^p}} \leq \| \hat{S}_{q-M_0} \bar{q}^{n+1} \circ \psi_q^{-1} \|_{L^2_{t,L^p}} + \sum_{j \geq q-M_0} \| \hat{\Delta}^j \bar{q}^{n+1} \circ \psi_q^{-1} \|_{L^2_{t,L^p}} \]
\[ \lesssim e^{CV(t)} (\| \hat{S}_{q-M_0} \bar{q}^{n+1} \|_{L^2_{t,L^p}} + \sum_{j \geq q-M_0} \| \hat{\Delta}^j \bar{q}^{n+1} \|_{L^2_{t,L^p}}) \]
by Lemma 2.8. Now, by Lemma 2.8 we have

\[ \| \hat{S}_{q-M_0} \bar{q}^{n+1} \|_{L^p} \lesssim \| J_{\psi_q}^{-1} \|_{L^\infty}^{\frac{1}{2}} (2^{-q \| \partial_x J_{\psi_q}^{-1} \|_{L^\infty}} \| J_{\psi_q} \|_{L^\infty} + 2^{-M_0} \| \partial_x \psi_q^{-1} \|_{L^\infty}) \| \theta_{q}^{n+1} \|_{L^p} \]

By Lemma 2.4 and Bernstein’s inequality, we estimate

\[ \| \hat{S}_{q-M_0} \bar{q}^{n+1} \|_{L^p} \lesssim \| J_{\psi_q}^{-1} \|_{L^\infty}^{\frac{1}{2}} (\| J_{\psi_q}^{-1} \|_{L^\infty} + 2^{-M_0} e^{CV(t)}) \| \theta_{q}^{n+1} \|_{L^p} \]

Taking \( L^2 \) norm over \([0, t]\) of this, we get

(9) \[ \| \hat{S}_{q-M_0} \bar{q}^{n+1} \|_{L^2_{t,L^p}} \lesssim e^{CV(t)} (e^{CV(t)} - 1 + 2^{-M_0}) \| \theta_{q}^{n+1} \|_{L^2_{t,L^p}} \]
For $|j-q| > 1$, we have due to compact support, $\hat{\Delta}_j \theta_{0,q} = 0$; hence, from (7) taking sum over $M_0 \geq q - j$, we have

$$\sum_{j \geq q-M_0} 2^{q(s+\alpha)}\|\hat{\Delta}_j \tilde{\theta}^{n+1}_q\|_{L^p_t L^p_x} \lesssim 2^{M_0(q_1 - e^{-\kappa t 2^{2q+1}})} \frac{1}{2} \sum_{j \geq q-M_0} 2^{q(s+\alpha)}\|\theta_{0,q}\|_{L^p_x}$$

+ $2^{q(s+\alpha)} 2^{M_0} e^{CV(t)} V^{1-\alpha}(t) \|\theta^{n+1}_q\|_{L^p_t L^p_x}$

+ $2^{M_0(1+\alpha)} \int_0^t 2^{q\alpha} e^{CV(\tau)} \|R_q\|_{L^p_x} d\tau$

Plugging (9) and (10) into (8) multiplied by $2^{q(s+\alpha)}$ gives

$$2^{q(s+\alpha)}\|\theta^{n+1}_q\|_{L^p_t L^p_x} \leq c(1 - e^{-\kappa t 2^{2q+1}}) \frac{1}{2} 2^{q\alpha}\|\theta_{0,q}\|_{L^p_x}$$

+ $c e^{CV(t)} \left(2^{-M_0} + 2^{M_0} V^{1-\alpha}(t)\right) 2^{q(s+\alpha)} \|\theta^{n+1}_q\|_{L^p_t L^p_x} + 2^{M_0(1+\alpha)} \int_0^t 2^{q\alpha} \|R_q\|_{L^p_x} d\tau$

We choose $M_0$ and $C_0 > 0$ such that $V(t) \leq C_0$ implies

$$c e^{CV(t)} \left(2^{-M_0} + 2^{M_0} V^{1-\alpha}(t)\right) \leq \frac{1}{2} \quad \text{and} \quad c e^{CV(t)} 2^{M_0(1+\alpha)} < 1$$

Then we have for $t \in [0, T]$, there exists $C_1 > 0$ such that

$$2^{q(s+\alpha)}\|\theta^{n+1}_q\|_{L^p_t L^p_x} \leq C_1 [(1 - e^{-\kappa t 2^{2q+1}}) \frac{1}{2} 2^{q\alpha}\|\theta_{0,q}\|_{L^p_x} + \int_0^t 2^{q\alpha} \|R_q\|_{L^p_x} d\tau]$$

As previously stated we take $s = \frac{1}{p} + 1 - 2\alpha$,

$$2^{q(\frac{1}{p} + 1 - \alpha)}\|\theta^{n+1}_q\|_{L^p_t B^{\frac{1}{p} + 1 - \alpha}_{p,1}} \leq C_1 [(1 - e^{-\kappa t 2^{2q+1}}) \frac{1}{2} 2^{q(\frac{1}{p} + 1 - \alpha)}\|\theta_{0,q}\|_{L^p_x} + \int_0^t 2^{q(\frac{1}{p} + 1 - \alpha)} \|R_q\|_{L^p_x} d\tau]$$

We sum over $q$ and by definition we have

$$\|\theta^{n+1}\|_{L^p_t B^{\frac{1}{p} + 1 - \alpha}_{p,1}} \leq \sum_{q \in \mathbb{Z}} C_1 [(1 - e^{-\kappa t 2^{2q+1}}) \frac{1}{2} 2^{q(\frac{1}{p} + 1 - \alpha)}\|\theta_{0,q}\|_{L^p_x} + \int_0^t 2^{q(\frac{1}{p} + 1 - \alpha)} \|R_q\|_{L^p_x} d\tau]$$

On $2^{q(\frac{1}{p} + 1 - \alpha)} \|R_q\|_{L^p_x}$ we use Lemma 2.6 with $\sigma = \frac{1}{p} + 1 - 2\alpha$, $N = 1$, $p = p_1$ to estimate; recall $R_q := (\hat{\Delta}_q - u^n - u^n)(\theta^{q+1}_q)_x - [\hat{\Delta}_q, u^n \partial_x] \theta^{q+1}_q$. 


\[
\sum_{|q' - q| \leq 4} 2^{q' - q} \frac{1}{t} + 2 - 2\alpha + \frac{q'}{p} \Phi(q') \frac{1}{L} \| \Delta_q u^n \|_{L^p} \| \hat{\Delta_q} \theta^{n+1} \|_{L^p}
\]

+ \sum_{q' \geq q - 3} 2^{-q(2\alpha) + \frac{q'}{p} + q' + q(\frac{1}{p} + 1)} \| \hat{\Delta_q} u^n \|_{L^p} \| \hat{\Delta_q} \theta^{n+1} \|_{L^p}

+ \sum_{q' \geq q - 3, |q' - q| \leq 1} 2^{(q - q')(\frac{1}{p} + 1 - 2\alpha + \min(\frac{1}{p}, \frac{1}{p}))} + \frac{q'}{p} + q''(\frac{1}{p} + 1 - 2\alpha)

(2^q \| \Delta_q u^n \|_{L^p} + 2^q \| \hat{\Delta_q} u^n \|_{L^p}) \| \hat{\Delta_q} \theta^{n+1} \|_{L^p}

Replacing the sum over \( q' \) where \( |q' - q| \leq 4 \) by constant multiples of the sum over \( q \) leads to

\[
\int_0^t 2^{t(\frac{1}{p} + 1 - \alpha)} \| R_q(t) \|_{L^p} \, dt \leq C \sum_q 2^{t(\frac{1}{p} + 1 - \alpha)} \| \hat{\Delta_q} u^n \|_{L^p} \| \hat{\Delta_q} \theta^{n+1} \|_{L^p}
\]

= \( C \| u^n \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \leq C \| \theta^n \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \)

by Hölder’s inequality and continuity of Hilbert transform. Thus,

\[
\| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \leq \sum_{q \in Z} C_1(1 - e^{-\kappa t 2^{q+1}}) \| \theta_{0,q} \|_{L^p} + C_2 \| \theta^n \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}}
\]

We go back to (6) and take \( L^1 \) norm over \([0,t]\) and multiply by \( 2^{q(1+2\alpha)} \) to obtain

\[
2^{q(1+1-2\alpha)} \| R_q \|_{L^p} \lesssim \sum_{q \in Z} 2^{q(1+2\alpha)} (1 - e^{-\kappa t 2^{q+1}}) \| \theta_{0,q} \|_{L^p} + \int_0^t e^{CV(t)} \| R_q \|_{L^p} \, dt
\]

A very similar procedure as before leads us to

\[
\| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} + \| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1}} \leq C_1 \sum_{q \in Z} (1 - e^{-\kappa t 2^{q+1}}) \| \theta_{0,q} \|_{L^p}
\]

+ \( C_2 \| \theta^n \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \| \theta^{n+1} \|_{L^2 B_{p,1}^{\frac{1}{p} + 1 - \alpha}} \)
We now estimate the last term for each $n$. For $n = 0$, we certainly have
\[ \|\theta^0\|_{L^2_t H^{\frac{1}{p}+1-\alpha}_{p,1}} = \|e^{-t\Lambda^{2\alpha}}\theta_0\|_{L^2_t H^{\frac{1}{p}+1-\alpha}_{p,1}} \lesssim \|\theta_0\|_{H^{\frac{1}{p}+1-2\alpha}_{p,1}} < \infty \]

On the other hand, for $\|\theta^1\|_{L^2_t H^{\frac{1}{p}+1-\alpha}_{p,1}}$, we apply Lemma 2.7 to
\[ \partial_t \theta^1 + u^0 \partial_x \theta^1 + \Lambda^{2\alpha} \theta^1 = 0; \quad \theta^1(x, 0) = \theta_0(x), \quad u^0 = H \theta^0 \]
i.e. (2) with $n = 1$, with $\rho = 2, r = 1, s = \frac{1}{p} + 1 - 2\alpha, p_1 = \infty$ to obtain
\[ \|\theta^1\|_{L^2_t H^{\frac{1}{p}+1-\alpha}_{p,1}} \lesssim e^{C_{T,0}^T \|\partial_x H \theta^0\|_{H^{0,\infty} \cap L^\infty}^\alpha dt} \|\theta_0\|_{H^{\frac{1}{p}+1-2\alpha}_{p,1}} \]

and
\[ \|\partial_x H \theta^0\|_{L^1_T (H^{0,\infty} \cap L^\infty)} \lesssim \|\theta^0\|_{L^1_T H^{\frac{1}{p}+1-\alpha}_{p,1}} < \infty \]

by Proposition 2.2(b) giving $\dot{B}^0_{p,1} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow \dot{B}^0_{\infty,\infty} \cap L^\infty$. Therefore, $C_T^2 \|\theta^0\|_{L^1_T H^{\frac{1}{p}+1-\alpha}_{p,1}} \|\theta^1\|_{L^1_T H^{\frac{1}{p}+1-\alpha}_{p,1}}$ is bounded. On the other hand, by Lebesgue dominated convergence theorem, we have
\[ \lim_{T \to 0^+} \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa T 2^{2(n+1)}})^\frac{1}{2} 2^{\theta(\frac{1}{p}+1-2\alpha)} \|\theta_{0,q}\|_{L^p} = 0 \]

Thus, we see that there exists some $\epsilon_0 > 0$ sufficiently small such that for all $t \leq T$ where
\[ T = \sup \{t > 0 : C_{T,0} \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa T 2^{2(n+1)}})^\frac{1}{2} 2^{\theta(\frac{1}{p}+1-2\alpha)} \|\theta_{0,q}\|_{L^p} \leq \epsilon_0 \} \]

we have
\[ \|\theta^1\|_{L^2_t H^{\frac{1}{p}+1-\alpha}_{p,1}} + \|\theta^1\|_{L^1_T H^{\frac{1}{p}+1-\alpha}_{p,1}} \leq 2\epsilon_0 \]

by (12). Thus, inductively for all $n \in \mathbb{Z}^+ \cup \{0\}, t \leq T$, we have if
\[ \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa T 2^{2(n+1)}})^\frac{1}{2} 2^{\theta(\frac{1}{p}+1-2\alpha)} \|\theta_{0,q}\|_{L^p} \leq \epsilon_0 \]

then
\[ \|\theta^{n+1}\|_{L^2_t H^{\frac{1}{p}+1-\alpha}_{p,1}} + \|\theta^{n+1}\|_{L^1_T H^{\frac{1}{p}+1-\alpha}_{p,1}} \leq 2\epsilon_0 \]

for all $\alpha \in (0, \frac{1}{2})$. We use $\dot{B}^0_{p,1} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow \dot{B}^0_{\infty,\infty} \cap L^\infty$ and obtain below as we did on (3),
\[ \|\theta^{n+1}\|_{L^\infty_T \dot{B}^{s+1-2\alpha}_{p,1}} \leq C e^{\int_0^T \|\partial_\tau u^n\|^2_{\dot{H}^{s+\infty}} \|\theta_0\|_{\dot{B}^{s+1-2\alpha}_{p,1}}} \leq C e^{\int_0^T \|\partial_\tau u^n\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \frac{d\tau}{\dot{B}^{s+1-2\alpha}_{p,1}}} \]

We bound above by

\[ C \int_0^T \|u^n\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \frac{d\tau}{\dot{B}^{s+1-2\alpha}_{p,1}} \leq C \|\theta_0\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \]

by Proposition 2.2 (a) of \( \Lambda^s \) being an isomorphism from \( \dot{B}^s_{p,m} \) to \( \dot{B}^{s-\alpha}_{p,m} \) and by continuity of Hilbert transform in \( \dot{B}^{s+1}_{p,1} \). Combining this and (14), we conclude that \((\theta^n)_{n\in\mathbb{N}} \) is uniformly bounded in \( \dot{L}^\infty_T \dot{B}^{s+1-2\alpha}_{p,1} \cap L^1_T \dot{B}^{s+1}_{p,1} \).

3.3. **Step 3: Strong Convergence.** We prove \((\theta^n)_{n\in\mathbb{N}} \) is a Cauchy sequence in \( \dot{L}^\infty_T \dot{B}^{s+1-2\alpha}_{p,1} \). Let \((n,m)\in\mathbb{N}^2, n > m \) and \( \theta^n, \theta^m = \theta^n - \theta^m \). Now

\[
\begin{align*}
\partial_\tau \theta^{n+1,m+1} &= -u^n \partial_x \theta^{n+1,m+1} - u^n \partial_x \theta^{m+1} - 2\alpha \theta^{n+1,m+1} \\
\theta_0^{n+1,m+1}(x) &= \theta^{n+1,m+1}(x,0) = \theta^{n+1}(x,0) - \theta^m(x,0) = 0
\end{align*}
\]

We rely on Lemma 2.7 with \( p_1 = \infty, \rho = \infty, s = \frac{1}{p} + 1 - 2\alpha, \rho_1 = r = 1 \)

\[ \|\theta^{n+1,m+1}\|_{\dot{L}^\infty_T \dot{B}^{s+1-2\alpha}_{p,1}} \]
\[ \leq C e^{\int_0^T \|u^n\|_{\dot{B}^{s+1-2\alpha}_{p,1}} (\|\theta_0^{n+1,m+1}\|_{\dot{B}^{s+1-2\alpha}_{p,1}} + \|u^n \partial_x \theta^{m+1}\|_{L^1_T \dot{B}^{s+1}_{p,1}})} \]
\[ \leq C e^{\int_0^T \|u^n \partial_x \theta^{m+1}(\tau)\|_{\dot{B}^{s+1-2\alpha}_{p,1}} d\tau} \]

Using Proposition 2.2 (c) now, we get

\[ \|u^n \partial_x \theta^{m+1}\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \leq \|u^n\|_{L^\infty} \|\theta^{m+1}\|_{\dot{B}^{s+1-2\alpha}_{p,1}} + \|\partial_x \theta^{m+1}\|_{L^\infty} \|u^n\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \]
\[ \leq \|\theta^n\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \|\theta^{m+1}\|_{\dot{B}^{s+1-2\alpha}_{p,1}} \]

Substitute this into (16) and obtain

\[ \|\theta^{n+1,m+1}\|_{\dot{L}^\infty_T \dot{B}^{s+1-2\alpha}_{p,1}} \leq C \|\theta^n\|_{\dot{L}^\infty_T \dot{B}^{s+1-2\alpha}_{p,1}} e^{\int_0^T \|\theta^{m+1}(\tau)\|_{\dot{B}^{s+1-2\alpha}_{p,1}} d\tau} \]

By (13) and (14), we can choose \( \epsilon_0 \) small enough such that
\[ \|\theta^{n+1,m+1}\|_{L^\infty_T \dot{B}^{1-2\alpha}_{p,1}} \leq \epsilon \|\theta^{n,m}\|_{L^\infty_T \dot{B}^{1-2\alpha}_{p,1}} \]

with \( \epsilon < 1 \). By induction

\[ \|\theta^{n+1,m+1}\|_{L^\infty_T \dot{B}^{1-2\alpha}_{p,1}} \leq \epsilon^{m+1}\|\theta^{n,0}\|_{L^\infty_T \dot{B}^{1-2\alpha}_{p,1}} \leq C \epsilon^{m+1}\|\theta_0\|_{\dot{B}^{1-2\alpha}_{p,1}} \]

Thus, \( \{\theta^n\}_{n\in\mathbb{N}} \) is a Cauchy sequence in \( \dot{L}^\infty_T \dot{B}^{1-2\alpha}_{p,1} \). By completeness, \( \theta^n \) converges strongly to \( \theta \) in \( \dot{L}^\infty_T \dot{B}^{1-2\alpha}_{p,1} \). By Fatou’s Lemma and (15), we obtain \( \theta \in L^1_T \dot{B}^{1-2\alpha}_{p,1} \). Passing to the limit into the approximate equation, we get a solution to (1) in \( \dot{L}^\infty_T \dot{B}^{1-2\alpha}_{p,1} \cap L^1_T \dot{B}^{1-2\alpha}_{p,1} \).

3.4. Step 4: Uniqueness. Let \( \vartheta^1 \) and \( \vartheta^2 \) be two solutions of (1) with same initial data in \( \dot{L}^\infty_T \dot{B}^{1-2\alpha}_{p,1} \cap L^1_T \dot{B}^{1-2\alpha}_{p,1} \). We can consider

\[ \partial_t \vartheta^{1,2} + u^1 \partial_x \vartheta^{1,2} + \Lambda^{2\alpha} \vartheta^{1,2} = -u^{1,2} \partial_x \vartheta^{2}, \quad \vartheta^{1,2}_0(x) = \vartheta^{1,2}(0, x) = 0 \]

We split two different cases. First, assume \( \alpha = \frac{1}{2} \). With \( -u^{1,2} \partial_x \vartheta^2 = f, \rho = \infty, \rho_1 = 1, \rho_1 = p, r = 1, s = \frac{1}{p} \) to obtain

\[ \|\vartheta^{1,2}\|_{L^\infty_T \dot{B}^{\frac{1}{p}}_{p,1}} \leq C e \int_0^T \|\partial_x u^1(t)\|_{\dot{B}^{\frac{1}{p}}_{p,1} \cap L^\infty} dt \|\vartheta^{1,2}_0\|_{\dot{B}^{\frac{1}{p}}_{p,1}} + \|u^{1,2} \partial_x \vartheta^2\|_{L^1_T \dot{B}^{\frac{1}{p}}_{p,1}} \]

\[ \leq C e \int_0^T \|\partial_x \vartheta^2(t)\|_{\dot{B}^{\frac{1}{p}}_{p,1}} dt \]

By continuity of Hilbert transform in \( \dot{B}^{\frac{1}{p}}_{p,1} \), we have

\[ \|\vartheta^{1,2}\|_{L^\infty_T \dot{B}^{\frac{1}{p}}_{p,1}} \leq C e \|\vartheta\|_{L^1_T \dot{B}^{\frac{1}{p}+1}_{p,1}} \int_0^T \|\vartheta^{1,2}(t)\|_{\dot{B}^{\frac{1}{p}}_{p,1}} dt \]

By Gronwall’s inequality, we obtain \( \|\vartheta^{1,2}(t)\|_{L^\infty_T \dot{B}^{\frac{1}{p}}_{p,1}} \equiv 0 \) for all \( 0 \leq t \leq T \).

Therefore, \( \vartheta^1 = \vartheta^2 \).

Now, for \( \alpha \in (0, \frac{1}{2}) \), suppose we have \( \vartheta_1, \vartheta_2 \) as two solutions with initial data in \( \dot{L}^\infty_T \dot{B}^{0}_{p,1} \cap L^1_T \dot{B}^{1}_{p,1} \). Note we have \( \dot{L}^\infty_T \dot{B}^{1-2\alpha}_{p,1} \cap L^1_T \dot{B}^{1-2\alpha}_{p,1} \subseteq \dot{L}^\infty_T \dot{B}^{0}_{p,1} \cap L^1_T \dot{B}^{1}_{p,1} \) for all \( p \in [1, 2\alpha, \infty] \). We apply Lemma 2.7 with \( p = \rho = \infty, r = 1, s = 0, p_1 = \infty, \rho_1 = 1 \).
\[ \|\theta^{1,2}\|_{L_T^\infty B_{\infty,1}^0} \leq C e^{C f_0^T \|\partial_x u(t)\|_{B_{\infty,\infty}^2} + \|u^{1,2}\|_{B_{\infty,1}^0} + \|\partial_x\theta^2\|_{L_T^1 B_{\infty,1}^0}} \cdot \int_0^T \|\theta^{1,2}\|_{L_T^\infty B_{\infty,1}^0} \|\theta^2(t)\|_{B_{\infty,1}^1} dt \]

due to Proposition 2.2(b). By Gronwall’s inequality, we obtain \[ \|\theta^{1,2}(t)\|_{L_T^\infty B_{\infty,1}^0} \equiv 0 \] for all \( t \in [0, T] \) which gives the desired result.

3.5. **Step 5: Smoothing Effect.** For the case \( \alpha = \frac{1}{2} \), we show that for all \( \beta \in \mathbb{R}^+ \), \( C_\beta = C(\beta), p \in [1, \infty), \)

\[ \|t^\beta\theta\|_{L_T^p B_{p,1}^{\frac{1}{p}+\beta}} \leq C_\beta e^{C f_0^T \|\theta\|_{B_{p,1}^{\frac{1}{p}+1}}} \|\theta\|_{L_T^p B_{p,1}^{\frac{1}{p}}}. \]

Note

\[ \partial_t(t^\beta \theta) = \beta t^{\beta-1} \theta + t^\beta \partial_t \theta = -u \partial_x(t^\beta \theta) - \Lambda(t^\beta \theta) + \beta t^{\beta-1} \theta; \quad (t^\beta \theta)(x, 0) = 0 \]

When \( \beta = 1 \), by Lemma 2.7 again with \( \rho = \rho_1 = p_1 = \infty, r = 1, s = \frac{1}{p}+1, \)

\[ \|t^\beta \theta\|_{L_T^p B_{p,1}^{\frac{1}{p}+1}} \leq C e^{C f_0^T \|\theta\|_{B_{p,1}^{\frac{1}{p}+1}}} \|\theta\|_{L_T^p B_{p,1}^{\frac{1}{p}}}. \]

Assume true for \( n \). Let \( \beta = n + 1 \): on

\[ \partial_t(t^{n+1} \theta) = -u \partial_x(t^{n+1} \theta) - \Lambda(t^{n+1} \theta) + \beta t^n \theta \]

with Lemma 2.7 again with \( \rho = \rho_1 = p_1 = \infty, r = 1, s = \frac{1}{p} + n + 1, p_1 = \infty, \rho_1 = \infty, \) we get

\[ \|t^{n+1} \theta\|_{L_T^p B_{p,1}^{\frac{1}{p}+n+1}} \leq C e^{C f_0^T \|\theta\|_{B_{p,1}^{\frac{1}{p}+1}}} \beta t^n \theta \|_{L_T^p B_{p,1}^{\frac{1}{p}+n}} \]

by Proposition 2.2(b) and the induction hypothesis. For general \( \beta \in \mathbb{R}^+ \), interpolation completes the proof.

3.6. **Blowup Criterion.** Let \( T^* \) be the maximum local existence time of \( \theta \)
in \( L_T^\infty B_{p,1}^{\frac{1}{p}+1} \cap L_T^1 B_{p,1}^{\frac{1}{p}+1} \). If \( T^* < \infty \), then we show \( \int_0^{T^*} \|\partial_x H \theta(t)\|_{L^\infty} dt = \infty \). Suppose \( \int_0^{T^*} \|\partial_x H \theta(t)\|_{L^\infty} dt < \infty \). By Lemma 2.7 with \( s = \frac{1}{p} + 1 - 2\alpha, r = 1, p_1 = \rho = \infty, \) for all \( t \in [0, T^*] \) we have

\[ \int_0^{T^*} \|\partial_x H \theta(t)\|_{L^\infty} dt < \infty \]
\( \| \theta(t) \|_{B^{\frac{1}{p}+1-2\alpha}} \leq M_{T^*} = Ce^{C \int_0^{T^*} \| \partial_t H \theta(t) \|_{L^\infty} dt} \| \theta_0 \|_{B^{\frac{1}{p}+1-2\alpha}} < \infty \)

Using \( \epsilon_0 \) from (13), we take \( \tilde{T} > 0 \) such that

\( \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa \tilde{T} 2^{q(2\alpha+1)}})^{\frac{1}{2}} M_{T^*} \leq \epsilon_0 \)

By (17) and (18) we have for all \( t \in [0, T^*) \),

\( \sum_{q \in \mathbb{Z}} (1 - e^{-\kappa \tilde{T} 2^{q(2\alpha+1)}})^{\frac{1}{2}} 2^{q(\frac{1}{p}+1-2\alpha)} \| \Delta_q \theta(t) \|_{L^p} \leq \epsilon_0 \)

Thus, by local existence result already obtained, we see that there exists solution \( \tilde{\theta}(t) \) on \([0, \tilde{T})\) to (1) with initial data \( \theta(T^* - \tilde{T}) \). By uniqueness of local solution, \( \tilde{\theta}(t) = \theta(t + T^* - \tilde{T}) \) on \([0, \frac{T^*}{2})\). Thus, \( \tilde{\theta} \) extends the solution \( \theta \) beyond \( T^* \). This completes the proof.

4. PROOF OF GLOBAL RESULT

4.1. Critical Case. We extend local results previously obtained globally in time via the method of modulus of continuity (MOC) introduced in [15] in the context of the critical QG. As this is discussed in [11] and [14] in detail, we only provide a sketch of the proof. A MOC is a function \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) that is continuous, increasing and concave such that \( \omega(0) = 0 \). We say a function \( \theta \) has a MOC \( \omega \) if for all \( x, y \in \mathbb{R} \), \(|\theta(x) - \theta(y)| \leq \omega(|x - y|) = \omega(\xi) \).

It is easy to show that the sup norm of the derivative of \( \theta \) is uniformly bounded if \( \theta \) has a MOC \( \omega \) for all time \( t > 0 \). The blow-up criterion which applies to our case, namely that if \( T^* \) is the first finite blow-up time, then

\[ \int_0^{T^*} \| \theta_x(\cdot, t) \|_{L^\infty} dx = \infty \]

was shown in [11]. This implies that in order to extend our local result, it suffices to show that \( \theta \) has a MOC \( \omega \) that is unbounded for all time \( t > 0 \); we note that in the critical case the rescaling of \( \omega \) would be \( \omega_\lambda(\xi) = \omega(\lambda \xi) \).

The following result is from [15]:

**Lemma 4.1.** If \( \theta(x, t) \) has a strict MOC \( \omega \) up to time \( T > 0 \) but not after \( T \), then at time \( T \), there exists \( x \neq y \) such that \( \theta(x, T) - \theta(y, T) = \omega(|x - y|) \).

Thus, we may assume that \( \theta \) loses a MOC \( \omega \) and show \( \frac{\partial}{\partial \eta} (\theta(x, T) - \theta(y, T)) < 0 \) to reach a contradiction. The estimate on the dissipation term modulus some constants is as follows:

\[ \int_0^{\frac{T}{2}} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^2} d\eta + \int_{\frac{T}{2}}^{\infty} \frac{\omega(\xi + 2\eta) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^2} d\eta \]
and we see that it implies the existence of $k > \eta$. Deduce that for small initial data $Z$ does not blow up; i.e. there exists $\theta$ the Hilbert transform of $Z$.

4.2. Supercritical Case. By Lemma 2.7, we see that it suffices to bound $Z(t) = \int_0^T \|\partial_x u(t)\|_{B_{\rho,1}^{1,1} \cap L^\infty} dt$. We choose $p_1 = \infty$ and by continuity of the Hilbert transform in homogeneous Besov spaces we obtain

\[
Z(t) \leq C \int_0^T \|u(t)\|_{B_{\rho,1}^{1,1} \cap W^{1,\infty}} dt \leq C \int_0^T \|u(t)\|_{B_{\rho,1}^{1,1}} dt \leq C \int_0^T \|\theta(t)\|_{B_{\rho,1}^{1,1}} dt.
\]

By Proposition 2.1. Applying lemma 2.7 on (1) with $p_1 = p, \rho = r = 1, s = \frac{1}{p} + 1 - 2\alpha$ and we satisfy $s = \frac{1}{p} + 1 - 2\alpha \leq 1 + \frac{1}{p}$ for all $\alpha \in (0, \frac{1}{2}), s > -N\min(\frac{1}{p}, \frac{1}{p'})$ to obtain

\[
\|\theta\|_{L^1_t B_{\rho,1}^{1,1}} \leq C e^{CZ(T)} \|\theta_0\|_{B_{\rho,1}^{1,1}}.
\]

Since $\rho = r$ we have $\|\theta\|_{L^1_t B_{\rho,1}^{1,1}} \leq \|\theta\|_{L^1_t B_{\rho,1}^{1,1}}$ and hence

\[
Z(t) \leq C \int_0^T \|\theta(t)\|_{B_{\rho,1}^{1,1}} dt \leq C \|\theta\|_{L^1_t B_{\rho,1}^{1,1}} \leq C e^{CZ(T)} \|\theta_0\|_{B_{\rho,1}^{1,1}}.
\]

Since the function $Z(t)$ depends continuously in time and $Z(0) = 0$, we deduce that for small initial data $Z$ does not blow up; i.e. there exists $\eta > 0$ and $C > 0$ such that $\|\theta_0\|_{B_{\rho,1}^{1,1}} < \eta \Rightarrow Z(t) \leq C \|\theta_0\|_{B_{\rho,1}^{1,1}}$.

5. Appendix

5.1. A: Subcritical Index. The key part of the proof that we must consider separately for the subindex is the uniform bound. Let $s > \frac{1}{p} + 1 - 2\alpha$ and we see that it implies the existence of $k > 1$ such that $s \geq \frac{1}{p} + 1 - \frac{2\alpha}{k}$. Using $\|\partial_x u^\alpha\|_{L^1_t B_{\rho,1}^{1,1} \cap L^\infty} \lesssim \|\theta^\alpha\|_{L^1_t B_{\rho,1}^{1,1}}$, by Lemma 2.7 on (2) with $n$ and $\rho = k$, $r = 1$, $s = \frac{1}{p} + 1 - \frac{2\alpha}{k}, p_1 = \infty$, we obtain

\[
\|\theta^\alpha\|_{L^1_t B_{\rho,1}^{1,1}} \leq t^{\frac{1}{k}} \|\theta^\alpha\|_{L^1_t B_{\rho,1}^{1,1}} \lesssim t^{\frac{1}{k}} \|\theta_0\|_{B_{\rho,1}^{1,1} \cap L^\infty} e^{\frac{1}{k} \int_0^T \|\partial_x u^{\alpha n-1}\|_{B_{\rho,1}^{1,1} \cap L^\infty} d\tau} \leq t^{\frac{1}{k}} \|\theta_0\|_{B_{\rho,1}^{1,1}} e^{\|\theta^\alpha n-1\|_{L^1_t B_{\rho,1}^{1,1}}}
\]

where the first inequality is Hölder’s. Thus, there exists some $\tilde{c}, \tilde{c}_0 > 0$ such that for all $n \in \mathbb{Z}^+ \cup \{0\}$ the following holds:
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\[ \frac{1}{t} \| \theta_0 \|_{B_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq \tilde{c}_0 \Rightarrow \| \theta^n \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq \tilde{c} \]

Now if \( n = 0 \), then

\[ \| \theta^0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + 1}} = \| e^{-t \Lambda^{2\alpha}} \theta_0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq t \frac{\| \theta_0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \| e^{-t \Lambda^{2\alpha}} \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq C_1 t \frac{\| \theta_0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \| e^{-t \Lambda^{2\alpha}} \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq C_1 \tilde{c}_0 }{t^2} \]

for \( t \) small enough. Now we iterate from (19) with \( n = 1 \): by Hölder’s inequality and Lemma 2.7 as before,

\[ \| \theta^1 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + 1}} \leq C_2 t \frac{\| \theta_0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq C_3 \| \theta^0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} e^{-t \Lambda^{2\alpha}} \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq C_2 \tilde{c}_0 e^{C_3 \tilde{c}_0} \leq 2C' \tilde{c}_0 \]

and

\[ \| \theta^2 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + 1}} \leq t \frac{\| \theta^2 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + 1}} \leq C_2 t \frac{\| \theta_0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq C_3 \| \theta^0 \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} e^{-t \Lambda^{2\alpha}} \|_{L_t^1 \dot{B}_{p,1}^{\frac{1}{2} + \frac{3}{2p}}} \leq C_2 \tilde{c}_0 e^{C_3 \tilde{c}_0} \leq 2C' \tilde{c}_0 \]

where \( C' = \max \{ C_1, C_2 \} \). \( \tilde{c}_0 \) small enough satisfying \( \min \{ cC_3 \tilde{c}_0, e^{2C_3 \tilde{c}_0} \} \leq 2 \). Taking \( \tilde{C} = 2C' \eta \) concludes the iteration. Using (20) in place of (13) and (14) in the Strong Convergence proof, completes the case for the subcritical index \( s > \frac{1}{p} + 1 - 2\alpha \).

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