ERGODICITY OF THE TWO-DIMENSIONAL MAGNETIC BENARD PROBLEM

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Abstract. We study the two-dimensional magnetic Bénard problem with noise, white in time. We prove the well-posedness including the path-wise uniqueness of the generalized solution, and the existence of the unique invariant, and consequently ergodic, measure under random perturbation.

Keywords: Bénard problem, ergodicity, invariant measure, irreducibility, Krylov-Bogoliubov theorem, strong Feller

1. Introduction

The Bénard problem is concerned with the motion of a horizontal layer of viscous fluid heated from below and the magnetic Bénard problem similarly with the electrically conducting viscous fluid. Both problems have attracted much attention in the past (cf. [18, 38] for the Bénard problem and [19, 20, 33, 39, 40] for the magnetic Bénard problem). In particular, systems of stochastic partial differential equations for these problems have been studied intensively (cf. [2, 9, 10] for the Bénard problem and [3] for the magnetic Bénard problem). For such systems, the existence of a unique invariant measure, if proven, describes the statistical equilibrium to which the system approaches. In this manuscript, we show the existence and uniqueness of the invariant measure, which is consequently ergodic, for the magnetic Bénard problem.

2. Preliminaries and Statement of Main Results

We consider a spatial domain $D = (0, L) \times (0, 1), L > 0$. We denote $\partial_t \triangleq \frac{\partial}{\partial t}, \partial_i \triangleq \frac{\partial}{\partial x_i}, i = 1, 2$ and let $(e_1, e_2)$ represent the standard basis in $\mathbb{R}^2$. We let $u(x, t) = (u_1, u_2)(x, t), b(x, t) = (b_1, b_2)(x, t), p(x, t), \theta(x, t)$ be the velocity and magnetic vector fields, pressure and temperature scalar fields respectively. We furthermore denote by $\nu_1, \nu_2, \kappa$ the kinematic viscosity, magnetic diffusivity, thermal diffusivity respectively and $s = H^2 \nu_1 \nu_2$ where $H$ is the Hartman number. Finally we let $n_j(x, t), j = 1, 2$, be the Gaussian random fields, white noise in time, to be
elaborated more below. With these notations, we consider
\begin{align}
\partial_t u + (u \cdot \nabla) u - \nu_1 \Delta u + \nabla \left( \frac{8}{2} |b|^2 \right) - s(b \cdot \nabla) b & = \theta c_2 + n^1, \\
\partial_t b + (u \cdot \nabla) b - (b \cdot \nabla) u - \nu_2 \Delta b & = n^2, \\
\partial_t \theta + (u \cdot \nabla) \theta - \kappa \Delta \theta - u_2 & = 0,
\end{align}
(1a, 1b, 1c)

We define and subsequently state our main results. We let $y$ denote the solution by $A$ instead of $x$ where $c$ inequality consists of a constant $s$ hereafter we assume $\nu = 1$, write $\int f$ for $\int_0^1 f(x)dx$ and when an equality or an inequality consists of a constant $c$ that is of no significance, we write $A \approx B$, $A \lesssim B$ instead of $A = cB$, $A \leq cB$ respectively. For convenience, hereafter we frequently denote the solution by $y = (u, b, \theta)$. We now set up the standard functional setting and subsequently state our main results. We let $H_1 = H_2$

\[ \triangle \equiv \{ u \in (L^2(D))^2 : \nabla \cdot u = 0, u_1|_{\{x_1=0\}} = u_1|_{\{x_1=L\}}, u_2|_{\{x_2=0\}} = u_2|_{\{x_2=1\}} = 0 \}, \]

$H_3 = (L^2(D))^2$, and set $H \triangleq H_1 \times H_2 \times H_3$, endowed with its norm for $y^j = (u^j, b^j, \theta^j)$,

\[ (y^1, y^2) = \int y^1 \cdot y^2, \quad |y^j|^2 = (y^j, y^j), \quad j = 1, 2. \]

We define

\[ V_1 \equiv \{ u \in H_1 \cap (H^1(D))^2 : u|_{\{x_2=0\}} = u|_{\{x_2=1\}} = 0, u \text{ is } L\text{-periodic in } x_1 \}, \]

$V_2 = H_2 \cap (H^1(D))^2$, $V_3 = \{ \theta \in H^1(D) : \theta|_{\{x_2=0\}} = \theta|_{\{x_2=1\}} = 0, \theta \text{ is } L\text{-periodic in } x_1 \}$. We let $V \equiv V_1 \times V_2 \times V_3$, $A_1$ be the Stokes operator defined by its bilinear form of

\[ a_1(u^1, u^2) = \nu_1 \sum_{i=1}^2 \int \nabla u_i^1 \cdot \nabla u_i^2, \quad a_1(u^j, u^2) = \|u^j\|_{V_1}^2, \quad u^j, u^2 \in V_1, \]

similarly $A_2$ defined by its bilinear form of

\[ a_2(b^1, b^2) = \nu_2 \sum_{i=1}^2 \int \nabla b_i^1 \cdot \nabla b_i^2, \quad a_2(b^j, b^2) = \|b^j\|_{V_2}^2, \quad b^j, b^2 \in V_2, \]

$A_3$ generated by the Dirichlet form of

\[ a_3(\theta^1, \theta^2) = \kappa \int \nabla \theta^1 \cdot \nabla \theta^2, \quad a_3(\theta^1, \theta^2) = \|\theta^j\|_{V_3}^2, \quad \theta^j, \theta^2 \in V_3, \]

and finally $A$ be the operator defined by

\[ a(y^1, y^2) = \sum_{i=1}^2 \nu_1 \int \nabla u_i^1 \cdot \nabla u_i^2 + \nu_2 \int \nabla b_i^1 \cdot \nabla b_i^2 + \kappa \int \nabla \theta^1 \cdot \nabla \theta^2, \quad a(y^j, y^2) = \|y^j\|_{V}^2, \]
$j = 1, 2$. We denote by $V_1 = D(A_H^1), V_2 = D(A_H^2), V_3 = D(A_H^3)$ and $\forall l \equiv D(A^l), l \in \mathbb{R}$ (cf. Chapter II Section 2.1 [38]). We also define bilinear continuous mappings $B_i, i = 1, \ldots, 5$ to satisfy
\[
\langle B_1(u^1, u^2), u^3 \rangle = \int \langle u^1 \cdot \nabla \rangle u^2 \cdot u^3, \quad \langle B_2(b^1, b^2), u^3 \rangle = \int \langle b^1 \cdot \nabla \rangle b^2 \cdot u^3,
\]
\[
\langle B_3(u^1, b^2) \cdot b^3 \rangle = \int \langle u^1 \cdot \nabla \rangle b^2 \cdot b^3, \quad \langle B_4(b^3, u^2), b^3 \rangle = \int \langle b^1 \cdot \nabla \rangle u^2 \cdot b^3,
\]
\[
\langle B_5(u^1, \theta^2), \theta^3 \rangle = \int \langle u^1 \cdot \nabla \rangle \theta^2 \theta^3,
\]
so that for $y = (u, b, \theta)$, denoting
\[
B(y, y) \triangleq \begin{pmatrix}
B_1(u, u) - B_2(b, b) \\
B_3(u, b) - B_4(b, u) \\
B_5(u, \theta)
\end{pmatrix}, \quad R_y \triangleq \begin{pmatrix}
\theta e_2 & 0 \\
0 & u_2 \\
0 & 0
\end{pmatrix}, \quad n \triangleq \begin{pmatrix}
n_1 \\
n_2
\end{pmatrix},
\]
we may rewrite (1a), (1b), (1c) as
\[
\partial_t y + Ay + B(y, y) = R_y + n.
\]
(4)

We briefly remark on a distinctive feature of the Bénard and magnetic Bénard problems in comparison to the Navier-Stokes equations (NSE) and the magneto-hydrodynamics (MHD) system. Considering the case e.g. $n \equiv 0$ for the MHD system which is (4) at $R \equiv 0, \theta \equiv 0$, we see that there exists $\alpha' > 0$, for example $\alpha' = \min\{\nu_1, \nu_2\} > 0$, such that
\[
(Ay, y) + (Ry, y) \geq \alpha' ||y||^2_H;
\]
i.e. $A + R$ is coercive on $H^1(D)$, while in the case of the Bénard and the magnetic Bénard problems, such a property is valid only if $\min\{\nu_1, \nu_2, \kappa\} > 0$ is sufficiently large (see [38] Chapter III Section 3.5).

Concerning $w^j(t), j = 1, 2$, we let $\{g_i^j, \lambda_i^j\}_{i=1}^{\infty}, j = 1, 2, 3$ be the eigenvectors and their corresponding eigenvalues of $A_j$ respectively. Then we define $w^j(t) \triangleq \sum_{i=1}^{\infty} \lambda_i^j(t) g_i^j$,
\[
G^j w^j(t) \triangleq \sum_{i=1}^{\infty} \sigma_i^j(t) \beta_i^j(t) g_i^j, \quad j = 1, 2
\]
(5)

where $\{\sigma_i^j\}_{i=1}^{\infty}, j = 1, 2$ are noise intensities, $\{\beta_i^j\}_{i=1}^{\infty}, j = 1, 2$ are families of independent standard one-dimensional Brownian motions defined $\forall t$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with expectation with respect to $\mathbb{P}$ denoted by $E$ and filtration of $\mathcal{F}_t = \sigma\{w(s) - w(\tau) : \tau \leq s \leq t\}$ where $w \triangleq (w^1, w^2, 0)$. We note that $\{g_i^j\}_{i=1}^{\infty}, j = 1, 2$ forms an orthonormal basis for $H_j, j = 1, 2, 3$ respectively.

Now letting $G \triangleq (G^1, G^2, 0), G \in L(\mathbb{H})$, we define our solution:

**Definition 2.1.** The stochastic process $y = (u, b, \theta)$ is a generalized solution over time interval $[t_0, T]$ of (4) if $\mathbb{P}$-a.e. $\omega \in \Omega$,
\[
y(., \omega) \in C([t_0, T]; \mathbb{H}) \cap L^2(t_0, T; D(A_H^1) \times D(A_H^2) \times V_3),
\]
(6)
∀ KAZUO YAMAZAKI

Let us now recall that irreducibility in \( \mathcal{V} \) implies that the transition function
\[
P(t, x, \Gamma) \triangleq P(\{ y(t; x) \in \Gamma \})
\]
\( \forall \) non-empty open set \( \Gamma \subset \mathcal{V} \). Moreover, their associated Markovian semigroup \( \{ P_t \} \) defined by
\[
(P_t \psi)(x) \triangleq E[\psi(y(t; x))]
\]
is Feller if \( P_t(C_b(\mathcal{V})) \subset C_b(\mathcal{V}) \) and strong Feller if \( P_t(B_b(\mathcal{V})) \subset C_b(\mathcal{V}) \) \( \forall t \in (t_0, T) \) where \( B_b(\mathcal{V}), C_b(\mathcal{V}) \) are the sets of all real Borel functions that are bounded, continuous and bounded respectively. We define a dual semigroup \( P_t^* \) in \( Pr(\mathcal{H}) \), the space of probability measures on \( \mathbb{H} \), by
\[
\int_{\mathcal{H}} \psi d(P_t^* \mu) = \int_{\mathcal{H}} P_t \psi d\mu.
\]
If \( P_t^* \mu = \mu \), then \( \mu \in Pr(\mathcal{H}) \) is called an invariant measure for the dynamical system \( (Pr(\mathcal{H}), P_t^*) \). We now state our first result:

**Theorem 2.1.** Suppose for both \( j = 1, 2 \),
\[
\sigma_i^j \neq 0 \quad \forall i = 1, 2, \ldots
\]
and
\[
\sigma_i^j \leq \frac{C}{i^{\frac{1}{2}+\gamma_0}} \quad \text{for some } \gamma_0 > 0
\]
for \( i \)'s large enough. Then \( \forall y_{t_0} \triangleq (u, b, \theta)(t_0) \in \mathbb{H} \), there exists a unique process \( y \) that solves (7) as defined in Definition 2.1 such that in addition to the regularity of (6),
\[
u(\cdot, \omega) \in L^2(t_0, T; D(A_1^{\min(\frac{1}{4}+\gamma, \frac{1}{2})})) \cap L^4(t_0, T; D(A_1^2)),
\]
\[
b(\cdot, \omega) \in L^2(t_0, T; D(A_2^{\min(\frac{1}{4}+\gamma, \frac{1}{2})})) \cap L^4(t_0, T; D(A_2^2))
\]
\( \forall \gamma < \gamma_0, \text{P-a.e. } \omega \in \Omega \). Moreover, it is a Markov process. Finally, there exists an invariant measure for (4).

In order to prove the uniqueness of the invariant measure, it suffices to prove the irreducibility and strong Feller property ([8] and e.g. Theorem 4.2.1 [6]). Under a slightly more restrictive condition, this is possible.

**Theorem 2.2.** Let \( l \in \mathbb{Z}, l \geq 2 \). Suppose that for both \( j = 1, 2 \), \( \{ \sigma_i^j \}_{i=1}^\infty \) satisfy (9) and
\[
\frac{c}{i^{\frac{1}{2}} \gamma} \leq \sigma_i^j < \frac{C}{i^{\frac{1}{2}}}
\]
(11)
for $i$’s large enough. Then $\forall y_0 \triangleq (u_0,b_0,\theta_0) \in V^l$, there exists a unique process $y$ that solves (7) as defined in Definition 2.1 at $t_0 = 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $y \in C([0,T];V^l)$; moreover, it is a Markov process. Finally, there exists a unique invariant measure $\mu$ of (7) at $t_0 = 0$, supported in $V^l$ such that

$$\lim_{t \to \infty} P(t,x,\Gamma) = \mu(\Gamma)$$

$\forall x \in V^l, \Gamma \in \mathcal{B}(V^l)$.

**Remark 2.1.**

(1) As a consequence of uniqueness, the invariant measure claimed in Theorem 2.2 is ergodic (cf. Theorem 3.2.6 [6]). To the best of the author’s knowledge, both Theorems 2.1 and 2.2 represent new results on the magnetic Bénard problem. Moreover, although the existence of an invariant measure for the Bénard problem is shown in [10], not its uniqueness. As the magnetic Bénard problem when $b \equiv 0$ becomes the Bénard problem, Theorem 2.2 proves the uniqueness of the invariant measure obtained in [10] for the Bénard problem as well.

(2) The proofs of Theorems 2.1 and 2.2 follow the approach of [10, 12] closely. The proof is also inspired by the previous work on ergodicity of the NSE, Burgers’ equation (see [5, 11, 14, 16, 17]); we also refer to [1, 22, 24, 25, 26, 31, 32, 34] as closely related important work. We remark however that in [10, 14], the path-wise uniqueness of the generalized solution was missing for a technical reason; in fact, in [14] Remark after Theorem 3.1, this uniqueness was stated as an open problem. We follow the approach of [13] to obtain such a path-wise uniqueness.

(3) For the generalization to $l$ non-integer and $l = 1$ cases, we refer to the Section 6 of [12]; we chose not to pursue this direction of research for simplicity of presentation.

3. **Proof of Theorem 2.1: Existence**

We consider two Ornstein-Uhlenbeck processes: for $\alpha \geq 0, j = 1, 2$, defined also for $t < 0$,

$$dz_{\alpha}^j + A_j z_{\alpha}^j(t)dt = -\alpha z_{\alpha}^j(t)dt + dB_j^j w^j,$$  \hspace{1cm} (12)

for which their solutions are of the form

$$z_{\alpha}^j(t) = \int_{-\infty}^{t} e^{(A_j - \alpha)(t-s)}G_j^j dw^j(s)$$  \hspace{1cm} (13)

(cf. [7]) and we collect their properties in the Appendix. By (10) we see that there exists $\gamma_0 > 0$ such that

$$\sum_{i=1}^{\infty} \frac{(\sigma_j^i)^2}{(\lambda_j^i)^{\frac{1}{2} + 2\gamma_0}} < \infty$$

because $\lambda_j^i \sim i$ as $i \to \infty$. Thus, by Lemma 5.2 (1) we see that $\mathbb{P}$-a.s.

$$z_{\alpha}^j \in C([t_0,T]; D(A_j^{\gamma_0 + \frac{1}{2}})),$$  \hspace{1cm} (14)

and by Lemma 5.1 (2)

$$G_j^j w^j \in C([t_0,T]; D(A_j^{\gamma_0 - \frac{1}{2}})).$$  \hspace{1cm} (15)
Now we define
\[ \pi := u - z^1_\alpha, \quad \bar{b} := b - z^2_\alpha \] (16)
so that by (4) and (12)
\[
\begin{align*}
\partial_t \pi + A_1 \pi + B_1(\pi, \bar{\pi}) + B_1(z^1_\alpha, \bar{\pi}) &+ B_1(z^1_\alpha, \pi) - B_2(\bar{\pi}, \bar{b}) - B_2(\pi, \bar{z}^2_\alpha) - B_2(z^2_\alpha, \bar{b}) \\
&= -B_1(z^1_\alpha, z^1_\alpha) + B_2(z^2_\alpha, z^1_\alpha) + \theta \varepsilon \alpha, \\
\partial_t \bar{b} + A_1 \bar{b} + B_3(\pi, \bar{b}) + B_3(z^1_\alpha, \bar{b}) - B_4(\bar{\pi}, \bar{b}) - B_4(\bar{\pi}, z^1_\alpha) - B_4(z^2_\alpha, \bar{b}) \\
&= -B_3(z^1_\alpha, z^1_\alpha) + B_4(z^2_\alpha, z^1_\alpha) + \alpha \varepsilon z^2_\alpha, \\
\partial_t \theta + A_3 \theta + B_5(\pi, \theta) &= -B_3(z^1_\alpha, \theta) + \pi \varepsilon + z^1_\alpha z^2_\alpha.
\end{align*}
\] (17a, 17b, 17c)
where \(z^1_\alpha, z^2_\alpha\) is the second component of \(z^1_\alpha\).

We now fix \(\omega \in \Omega\) and take \(L^2\)-inner products of (17a), (17b), (17c) with \((\pi, \bar{\pi}, \bar{b}, \theta)\) to obtain in sum
\[
\begin{align*}
\frac{1}{2} \partial_t \| (\pi, \bar{\pi}, \bar{b}, \theta) \|^2 &+ \| (\pi, \bar{\pi}, \bar{b}, \theta) \|^2 \\
= &\langle B_1(\pi, \bar{\pi}), z^1_\alpha \rangle - \langle B_2(\bar{\pi}, \bar{b}, \bar{\pi}), z^2_\alpha \rangle + (B_1(z^1_\alpha, \bar{\pi}), z^1_\alpha) - (B_2(z^2_\alpha, \pi), z^1_\alpha) \\
+ &\langle \theta \varepsilon \alpha, \bar{\pi} \rangle + \alpha \varepsilon (z^1_\alpha, z^1_\alpha) + (B_3(\pi, \bar{b}, \bar{b}), z^1_\alpha) \\
+ &\langle B_3(z^1_\alpha, \bar{b}), z^1_\alpha \rangle - \langle B_4(z^2_\alpha, \bar{b}), z^1_\alpha \rangle + \alpha \varepsilon (z^2_\alpha, \bar{b}) + \langle \pi \varepsilon \alpha, e \theta \rangle + (z^1_\alpha, e \theta) \\
\leq &\| \pi \|_{L^4} \| \bar{\pi} \|_{L^4} \| z^1_\alpha \|_{L^4} + \| \bar{\pi} \|_{L^4} \| \pi \|_{L^4} \| z^1_\alpha \|_{L^4} + \| z^1_\alpha \|_{L^4} \| \bar{\pi} \|_{L^4} + \| \bar{\pi} \|_{L^4} \| \pi \|_{L^4} \| z^1_\alpha \|_{L^4} + \| z^1_\alpha \|_{L^4} \| \bar{\pi} \|_{L^4} \| \pi \|_{L^4} \| z^1_\alpha \|_{L^4} \\
+ &\| z^2_\alpha \|_{L^4} \| \bar{\pi} \|_{L^4} \| z^2_\alpha \|_{L^4} + \| z^2_\alpha \|_{L^4} \| \bar{\pi} \|_{L^4} \| z^1_\alpha \|_{L^4} + \| \bar{\pi} \|_{L^4} \| \pi \|_{L^4} \| z^2_\alpha \|_{L^4} \\
+ &\| \theta \|_{L^4} \| \bar{\pi} \|_{L^4} + \| \pi \|_{L^4} \| \bar{\pi} \|_{L^4} \| \pi \|_{L^4} \| z^2_\alpha \|_{L^4} + \| \bar{\pi} \|_{L^4} \| \bar{\pi} \|_{L^4} \| z^1_\alpha \|_{L^4} \\
+ &\| \pi \|_{L^4} \| \bar{\pi} \|_{L^4} \| z^2_\alpha \|_{L^4} + \| z^1_\alpha \|_{L^4} \| \bar{\pi} \|_{L^4} + \| z^2_\alpha \|_{L^4} \| \pi \|_{L^4} \| z^1_\alpha \|_{L^4} \\
\leq &\left( \frac{1}{2} \langle (\pi, \bar{\pi}, \bar{b}, \theta)(t) \rangle + c((\pi, \bar{\pi}, \bar{b}, \theta)(t))^2 \right)(1 + \| \pi \|_{L^4} \| z^1_\alpha \|_{L^4} + \| z^2_\alpha \|_{L^4}) (18).
\end{align*}
\] where we used integration by parts, Hölder’s inequalities, Gagliardo-Nirenberg inequalities of e.g. \( |u|_{L^4(D)} \lesssim |u|_{L^3(D)}^\frac{1}{2} \|u\|_{L^3(D)}^\frac{1}{2} \) (cf. [28] Lemma 6.2, and more generally [30]), and Young’s inequalities. We also relied on important cancellations such as
\[
-\langle B_2(\bar{\pi}, \bar{b}), \bar{\pi} \rangle - \langle B_4(\bar{\pi}, \bar{b}), \bar{\pi} \rangle = 0, \\
-\langle B_3(z^1_\alpha, \bar{b}), \bar{\pi} \rangle - \langle B_4(z^2_\alpha, \bar{b}), \bar{\pi} \rangle = 0.
\]
After subtracting \(\frac{1}{2} \| (\pi, \bar{\pi}, \bar{b}, \theta) \|^2 \) from both sides of (18), we obtain by the Sobolev embedding of \( D(A_1^\frac{1}{2}) \hookrightarrow L^4(D) \) and (14) that \( \forall t \in [t_0, T] \)
\[
\| (\pi, \bar{\pi}, \bar{b}, \theta)(t) \|^2 + \int_{t_0}^T \| (\pi, \bar{\pi}, \bar{b}, \theta) \|^2 ds \lesssim \| (\pi, \bar{\pi}, \bar{b}, \theta)(t_0) \|^2 + 1 \lesssim 1.
\] (19)

**Proposition 3.1.** Under the hypothesis of Theorem 2.1, \( \alpha \geq 0 \) on \([t_0, T] \) such that \((u, b, \theta)(t_0) \in H \), there exists a unique solution \((\pi, \bar{\pi}, \bar{b}, \theta)(\cdot) \) to (17a), (17b), (17c) such that \((\pi(t_0), \bar{b}(t_0), \theta(t_0)) = (u(t_0) - z^1_\alpha(t_0), b(t_0) - z^2_\alpha(t_0), \theta(t_0)) \) and
\[
(\pi, \bar{\pi}, \bar{b}, \theta) \in C([t_0, T]; H) \cap L^2(t_0, T; V) \quad \mathbb{P} \text{ a.e. } \omega \in \Omega.
\] (20)
Moreover, the solution is independent of $\alpha$.

Proof. With the a priori estimates of (19), a standard Galerkin approximation scheme shows the existence of the solution (see [4, 28, 38]). We note that due to (16) the regularity of $u$ and $b$ are obtained from $\overline{u}, z^1_\alpha$ and $\overline{b}, z^2_\alpha$ respectively. This is the reason why in Definition 2.1, we require only $u \in L^2(t_0, T; D(A^1_T)), b \in L^2(t_0, T; D(A^1_T))$ instead of $L^2(t_0, T; V_1), L^2(t_0, T; V_2)$.

We now prove the uniqueness. Suppose there exists $(\overline{u}, \overline{b}, \theta^1), (\overline{u}, \overline{b}, \theta^2)$ that are both solutions. Then defining
\[
\delta \overline{u} \triangleq \overline{u}^1 - \overline{u}^2, \quad \delta \overline{b} \triangleq \overline{b}^1 - \overline{b}^2, \quad \delta \theta \triangleq \theta^1 - \theta^2,
\]
we see that from (17a), (17b), (17c)
\[
\begin{align*}
\partial_t \delta \overline{u} + A_1 \delta \overline{u} &= -B_1(\delta \overline{u}, \overline{u}^1) - B_1(\overline{u}^2, \delta \overline{u}) - B_1(\delta \overline{u}, z^1_\alpha) - B_1(z^1_\alpha, \delta \overline{u}) \quad (22a) \\
&+ B_2(\delta \overline{b}, \overline{b}^1) + B_2(\overline{b}^2, \delta \overline{b}) + B_2(\delta \overline{b}, z^2_\alpha) + B_2(z^2_\alpha, \delta \overline{b}) + \delta \theta e_2, \\
\partial_t \delta \overline{b} &+ A_2 \delta \overline{b} = -B_3(\delta \overline{b}, \overline{b}^1) - B_3(\overline{b}^2, \delta \overline{b}) - B_3(\delta \overline{b}, z^2_\alpha) - B_3(z^2_\alpha, \delta \overline{b}) \quad (22b) \\
&+ B_4(\delta \overline{b}, \overline{u}^1) + B_4(\overline{u}^2, \delta \overline{b}) + B_4(\delta \overline{b}, z^1_\alpha) + B_4(z^1_\alpha, \delta \overline{b}), \\
\partial_t \delta \theta + A_3 \delta \theta &= -B_5(\delta \theta, \theta^1) - B_5(\overline{u}^2, \delta \theta) - B_5(z^1_\alpha, \delta \theta) + \delta \theta e_2, \quad (22c)
\end{align*}
\]
with $\delta \overline{u}_2 = \overline{u}^1_2 - \overline{u}^2_2, \overline{u}^2_2$ being the second component of $\overline{u}^j, j = 1, 2$.

Taking $L^2$-inner products on (22a), (22b), (22c) with $(\delta \overline{u}, \delta \overline{b}, \delta \theta)$ respectively we obtain similarly to (18)
\[
\begin{align*}
\frac{1}{2} \partial_t (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) &+ \frac{1}{2} \partial_t (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) \nonumber \\
&= - \int (\delta \overline{u} \cdot \nabla) \overline{u}^1 \cdot \delta \overline{u} + \int (\delta \overline{u} \cdot \nabla) \overline{u} \cdot z^1_\alpha + \int (\delta \overline{b} \cdot \nabla) \overline{b}^1 \cdot \delta \overline{b} - \int (\delta \overline{b} \cdot \nabla) \overline{b} \cdot z^2_\alpha \\
&+ \int \delta \theta e_2 \cdot \delta \overline{u} - \int (\delta \overline{u} \cdot \nabla) \overline{b}^1 \cdot \delta \overline{b} + \int (\delta \overline{u} \cdot \nabla) \delta \overline{b} \cdot z^2_\alpha + \int (\delta \overline{b} \cdot \nabla) \overline{u} \cdot \delta \overline{b} \\
&- \int (\delta \overline{b} \cdot \nabla) \delta \overline{u} \cdot z^1_\alpha - \int (\delta \overline{b} \cdot \nabla) \delta \overline{b} \cdot z^2_\alpha \leq \|

\begin{align*}
\frac{1}{2} (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) &+ \frac{1}{2} \partial_t (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) \nonumber \\
&\leq \frac{1}{2} \left[ (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) + \frac{1}{2} \partial_t (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) \right] + 1
\end{align*}
\]
where we made the crucial cancellations of
\[
\int (\overline{b}^1 \cdot \nabla) \delta \overline{b} \cdot \delta \overline{u} + \int (\overline{b}^2 \cdot \nabla) \delta \overline{u} \cdot \delta \overline{b} = 0,
\]
\[
\int (z^1_\alpha \cdot \nabla) \delta \overline{b} \cdot \delta \overline{u} + \int (z^2_\alpha \cdot \nabla) \delta \overline{u} \cdot \delta \overline{b} = 0.
\]
Subtracting $\frac{1}{2} (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2)$ from both sides, we obtain
\[
\begin{align*}
\partial_t (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) &+ \frac{1}{2} (|\delta \overline{u}|^2 + |\delta \overline{b}|^2 + |\delta \theta|^2) \nonumber \\
&\leq |(\delta \overline{u}, \delta \overline{b}, \delta \theta)|^2 (\|\overline{u}^1_2, \theta^1\|_L^2 \quad \text{and} \quad (z^1_\alpha, z^2_\alpha) |_{L^2}^2 + 1).
\end{align*}
\]
Gronwall’s inequality with (14) and (19) implies uniqueness.

Concerning Markov property, it suffices to show that for $y = (u, b, \theta)$,

$$E[\psi(y(t); t_0, y(t_0))|F_s] = P_{t-s}(\psi)(y(s; t_0, y(t_0)))$$

$\forall \ t_0 < s < t, y(t_0) \in \mathbb{H}, \psi \in C_0(\mathbb{H})$. This result is classical and can be found in [7] Theorem 9.14.

We have already shown that the solution $(\overline{\pi}, \overline{b}, \theta)$ to (17a), (17b), (17c) is unique $\forall \alpha \geq 0$. For a fixed $\alpha \geq 0$, we let $(\pi^\alpha, \theta^\alpha)$ be the unique solution to (17a), (17b), (17c). Then by (16), $u_\alpha = \pi^\alpha \alpha z^3 + z^1 + \alpha z^3$ and for $\alpha \geq 0$, we have $(\pi^\alpha, \theta^\alpha, \theta) \in L^2(0, T; V)$. Moreover, for $j = 1, 2$, denoting by $z_j$, the solution to (12) at $\alpha = 0$, we see that they satisfy

$$\partial_t(z_j^\alpha - z_j^\beta) = -A_j(z_j^\alpha - z_j^\beta) - \alpha z_j^\beta$$

so that

$$(z_j^\alpha - z_j^\beta)(t) = e^{-(t-t_0)A}e^{-(t-t_0)A}(z_j^\alpha - z_j^\beta)(t_0) - \int_{t_0}^{t} e^{-(t-s)A}A\alpha z_j^\beta(s)ds$$

$\forall \alpha \geq 0$ and it is well-known (cf. [35]) that $(z_j^\alpha - z_j^\beta)(t) \in L^2(t_0, T; V_j)$. Thus, $u_\alpha - z_1 \in L^2(t_0, T; V_j), b_\alpha - z_2 \in L^2(t_0, T; V_2)$ and therefore $\forall \alpha \geq 0$, we obtain a unique solution $(u_\alpha, b_\alpha)$ independent of $\alpha$; it is also clear that $\theta$ is independent of $\alpha$.

\textbf{Proposition 3.2.} Under the hypothesis of Theorem 2.1, suppose that for some $\gamma \in (0, 1)$ and $\gamma_0 \in (0, 2\gamma_0)$, where $\gamma_0 \in (0, 1]$, $y(t_0) \in D(A^\gamma)$. Then $P$-a.e. $\omega \in \Omega$

$$y \in C([t_0, T]; D(A^\min(\gamma, \frac{1}{2} + \gamma_0))) \cap L^2([t_0, T]; D(A^\max(1, \frac{1}{2} + \gamma_0)))$$

Proof. For both $j = 1, 2$, $z_j^\alpha \subset C([t_0, T]; D(A_j^\gamma))$ by (14) and hence due to (16), it suffices to show that

$$(\pi, \overline{b}, \theta) \subset C([t_0, T]; D(A_\min(\gamma, \frac{1}{2} + \gamma_0)))$$

We first remark that $|A^\gamma_1| \equiv |A^\gamma_2|$. We take $L^2$-inner products on (17a), (17b), (17c) with $(A_1^\gamma \pi, A_2^\gamma \overline{b}, A_3^\gamma \theta)$ to obtain

$$\frac{1}{2} \int (A^\gamma_1 \pi, A^\gamma_2 \overline{b}, A^\gamma_3 \theta) = \int (\pi \cdot \nabla) \pi + (\pi \cdot \nabla) z^1 - (\overline{b} \cdot \nabla) \overline{b} - (\overline{b} \cdot \nabla) z^2 - (\alpha z^1 \cdot \nabla) \cdot A_1^\gamma \pi$$

$$- \int (z^1 \cdot \nabla) z^1 - (z^2 \cdot \nabla) z^2 - \theta \epsilon^2 - \alpha z^1 \cdot A_1^\gamma \pi$$

$$- \int (\pi \cdot \nabla) \overline{b} + (\pi \cdot \nabla) z^1 - (\overline{b} \cdot \nabla) \overline{b} - (\overline{b} \cdot \nabla) z^2 - (\alpha z^1 \cdot \nabla) \overline{b}$$

$$- \int (z^1 \cdot \nabla) z^2 - (z^2 \cdot \nabla) z^1 - \alpha z^1 \cdot A_2^\gamma \overline{b}$$

$$- \int (\pi \cdot \nabla) \theta + (z^1 \cdot \nabla) \theta + \overline{b} + z^1 A_3^\gamma \theta$$.

We bound e.g.,

$$\int (\pi \cdot \nabla) \pi \cdot A_1^\gamma \pi \leq |A_1^\gamma| \int (\pi \cdot \nabla) ||A_1^\gamma|| \leq |A_1^\gamma| \frac{1}{2} ||A_1^\gamma||^2 \leq |A_1^\gamma| \frac{1}{2} ||A_1^\gamma||^2$$
by Hölder’s inequality and Lemma 5.3. We compute similarly on other nonlinear terms to obtain

\[
\frac{1}{2} \partial_t |(A_1^\gamma \Pi, A_2^\gamma \Pi, A_3^\gamma \theta)|^2 + |(A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta)|^2 \leq \left( |(A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta)|^2 + |(A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta)|^2 \right) \leq |(A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta)|^2 \leq \left( |(A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta)|^2 + |(A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta)|^2 \right)
\]

(25)

Thus, by Gronwall’s inequality we obtain

\[
|A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta(t)|^2 + \int_{t_0}^t |A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta|^2 ds \leq |A_1^{\frac{1}{2} + \gamma} \Pi, A_2^{\frac{1}{2} + \gamma} \Pi, A_3^{\frac{1}{2} + \gamma} \theta(t_0)|^2 e^{\int_{t_0}^t 1 + \|\Pi, \Pi, \theta\|_V^2} ds.
\]

The computation above may be repeated for \( \gamma \) replaced by \( \min\{\gamma, \frac{1}{4} + \gamma_0\} \); therefore,

\[
|A_1^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_2^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_3^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \theta(t)|^2 + \int_{t_0}^t |A_1^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_2^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_3^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \theta|^2 ds \leq |A_1^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_2^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_3^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \theta(t_0)|^2 e^{\int_{t_0}^t 1 + \|\Pi, \Pi, \theta\|_V^2} ds.
\]

Finally, as \( (\Pi, \Pi, \theta)(t_0) = (u - z_0^1, b - z_0^2, \theta)(t_0) \) due to (16) where \( (u, b, \theta)(t_0) \in D(A^\gamma) \) by hypothesis and \( z_0^\gamma \in C([t_0, T]; D(A_0^{\frac{1}{4} + \gamma})) \) by (14), we see that

\[
|A_1^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_2^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \Pi, A_3^{\min\{\gamma, \frac{1}{4} + \gamma_0\}} \theta(t_0)|^2 < \infty.
\]

Thus, due to (20), the proof is complete. \( \square \)

**Proposition 3.3.** (Dissipativity) Let \( y(t, t_0, (0, 0, \theta(t_0)), \omega) \) be the solution \( y = (u, b, \theta)(\cdot, \omega) \) at time \( t \) that has values \( (u, b, \theta)(t_0) = (0, 0, \theta(t_0)) \) in \( \mathbb{H} \). Suppose that
−1 \leq \theta(t_0) \leq 1 \ a.e. \ x \in D. \ Then \ under \ the \ hypothesis \ of \ Theorem \ 2.1, \ there \ exists \ a \ random \ variable \ r(\omega) \ P-a.s. \ finite \ such \ that \ for \ \gamma \in (0, \frac{1}{4}) \cap (0, 2\gamma_0]

\sup_{-\infty < t_0 \leq 0} |A^\gamma y(0, t_0, y(t_0), \omega)| \leq r(\omega). \quad (27)

Proof. We take \( L^2\)-inner products on (17a), (17b) with \((\pi, \delta)\) to obtain similarly to (18), applying Hölder’s, Gagliardo-Nirenberg and Young’s inequalities,

\begin{align*}
\frac{1}{2} \partial_t |(\pi, \delta)|^2 + \|\pi\|_{V_1}^2 + \|\delta\|_{V_2}^2 \\
\leq \frac{1}{2} \|\pi\|_{V_1}^2 + \|\delta\|_{V_2}^2 + \frac{\epsilon}{2} |(\pi, \delta)|^2 + c|\pi|^2((z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4})^4 + |\theta(z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4}^4 + |\omega(z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4}^4
\end{align*}

for \( \epsilon > 0 \) arbitrary small. We may choose \( \epsilon > 0 \) sufficiently small so that by Poincare’s inequality,

\begin{align*}
\epsilon |(\pi, \delta)|^2 + \frac{1}{2} \|\pi\|_{V_1}^2 + \|\delta\|_{V_2}^2 \leq \|\pi\|_{V_1}^2 + \|\delta\|_{V_2}^2.
\end{align*}

Thus, subtracting \( \frac{1}{2} \|\pi\|_{V_1}^2 + \|\delta\|_{V_2}^2 \) from both sides of (28), we may obtain for some constant \( c_0 > 0 \)

\begin{align*}
|(\pi, \delta)(t_0)|^2 - |(\pi, \delta)(t_0)|^2 e^{-\epsilon + c_0((z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4})^4} &
\leq \epsilon \int_{t_0}^t |(z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4}^2 + |(\theta(z^{\alpha_1}_0, z^{\alpha_2}_0))_{L^4}|^2 e^{\epsilon - \epsilon + c_0((z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4})^4} ds.
\end{align*}

Now \( z^{\alpha_0}_0(t) \) is an ergodic process with values in \( (L^4(D))^2 \); thus,

\begin{align*}
l \lim_{t_0 \to \infty} - \frac{1}{t_0} \int_{0}^{t_0} |z^{\alpha_0}_0|_{L^4}^4 ds = E[|z^{\alpha_0}_0(0)|_{L^4}]\]

while by Sobolev embedding of \( D(A^{\frac{1}{2}}) \hookrightarrow L^4(D) \) and Lemma 5.2 (2), \( \forall \ s \), we know

\begin{align*}
\lim_{a \to \infty} E[|z^{\alpha}_0(s)|_{L^4}] = 0
\end{align*}

so that for \( c_0 > 0 \) fixed, we have

\begin{align*}
- \epsilon + c_0 E[|z^{\alpha_0}_0(0)|_{L^4}]^4 \leq -\frac{\epsilon}{2}
\end{align*}

for \( \alpha > 0 \) sufficiently large. Thus,

\begin{align*}
l \lim_{t_0 \to \infty} - \frac{1}{t_0} \int_{0}^{t_0} - \epsilon + c_0(z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4}^4 ds \leq -\frac{\epsilon}{2}.
\end{align*}

This implies that for given \( \omega \in \Omega \) for some \( \tau(\omega) < 0 \) sufficiently small, we have

\begin{align*}
\sup_{t_0 < \tau(\omega)} \int_{t_0}^{0} - \epsilon + c_0(z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4}^4 ds \leq (-\frac{\epsilon}{4})(-t_0) = (\epsilon)(t_0). \quad (30)
\end{align*}

Moreover, by (14) and the Sobolev embedding of \( D(A^{\frac{1}{2}}) \hookrightarrow (L^4(D))^2 \), there exists a constant \( c(\omega) \) such that

\begin{align*}
\sup_{\tau(\omega) < 0 \leq t_0 \leq 0} \int_{t_0}^{0} - \epsilon + c_0(z^{\alpha_1}_0, z^{\alpha_2}_0)|_{L^4}^4 ds \leq c(\omega).
\end{align*}
Thus, by (30) and (31) we have the bound of
\[
\int_{t_0}^t -\epsilon + c_0 \|(z_{\alpha_t}^1, z_{\alpha_t}^2)\|_{L^4}^4 \, ds \leq \begin{cases} 
\frac{\epsilon}{4} (t_0) & \text{if } t_0 < \tau(\omega), \\
\epsilon c(\omega) & \text{if } \tau(\omega) \leq t_0,
\end{cases}
\]
\(\Delta \tau(t_0, \omega).\)

We observe that for a fixed \(\tau(\omega)\), \(\lim_{t_0 \to -\infty} \tau(t_0, \omega) = -\infty\). Besides, for \(\tau \leq -1, \forall t \in [-1, 0]\), we also have
\[
\int_{t_0}^t -\epsilon + c_0 \|(z_{\alpha_t}^1, z_{\alpha_t}^2)\|_{L^4}^4 \, ds \leq \tau(t_0, \omega) + c(\omega).
\]

Under our hypothesis, \(z_{\alpha}^i\) is a stationary process with values in \((L^4(D))^2\); thus, for all \(t < 0\),
\[
\|z_{\alpha}^i(t, \omega)\|_{L^4} \leq c(\omega)(1 + |t|), \quad |z_{\alpha}^i(t, \omega)| \leq c(\omega)(1 + |t|) \quad (32)
\]
for some \(c(\omega) > 0\) taken larger than before if necessary. Thus, \(\forall t \in [-1, 0], t_0 \leq t\), from (29),
\[
|(\pi, \bar{b})(t)|^2 \leq |(\pi, \bar{b})(t_0)|^2 e^{\tau(t_0, \omega) + c(\omega)} + \int_{t_0}^t e^{\tau(s, \omega) + c(\omega)} c(\omega)(1 + |s|) \, ds \leq r_1(\omega) \quad (33)
\]
for some \(r_1(\omega)\) due to Proposition 5.4.

We go back to (28); subtracting \(\frac{1}{2}(\|\pi\|_{\chi_1}^2 + \|\bar{b}\|_{\chi_2}^2)\) from both sides of (28) leads to
\[
\partial_t |(\pi, \bar{b})|^2 + \|\pi\|_{\chi_1}^2 + \|\bar{b}\|_{\chi_2}^2 \leq |(\pi, \bar{b})|^2(1 + \|z_{\alpha}^1, z_{\alpha}^2\|_{L^4}^4) + \|z_{\alpha}^1, z_{\alpha}^2\|_{L^4}^4 + |\theta|^2.
\]
Integrating over \([-1, 0]\) leads to
\[
\int_{-1}^0 |\pi|_{\chi_1}^2 + |\bar{b}|_{\chi_2}^2 \, ds \leq r_2(\omega) \quad (34)
\]
for some \(r_2(\omega)\) due to (32), (33) and Proposition 5.4. Next, taking \(L^2\)-inner products of (17c) with \(\theta\), applying Young’s inequality and integrating over \([-1, 0]\) give
\[
\int_{-1}^0 |\theta|_{\chi_3}^2 \, ds \leq \frac{|\theta(-1)|^2}{2} + \int_{-1}^0 |\pi|^2 + \|\bar{b}\|^2 + \|z_{\alpha}^1\|^2_2 \, ds \leq r_3(\omega) \quad (35)
\]
for some \(r_3(\omega)\) due to Proposition 5.4, (32) and (33).

Next, we may compute from (25),
\[
\frac{1}{2} \partial_t |(A_1^{\frac{1}{2} + \gamma} \pi, A_2^{\frac{1}{2} + \gamma} \bar{b})|^2 \leq |(A_1^{\frac{1}{2} + \gamma} \pi, A_2^{\frac{1}{2} + \gamma} \bar{b})|^2 + c(1 + |(A_1^{\frac{1}{2} + \gamma} \pi, A_2^{\frac{1}{2} + \gamma} \bar{b})|^4 + |(A_1^{\frac{1}{2} + \gamma} \pi, A_2^{\frac{1}{2} + \gamma} \bar{b})|^4 + |\theta|_{\chi_3}^2).
\]
Subtracting \(|(A_1^{\frac{1}{2}+\gamma}u, A_2^{\frac{1}{2}+\gamma}v)|^2\) from both sides of (36), using that \(\frac{1}{4} + \frac{k}{2} \leq \gamma_0 + \frac{1}{4}\) and Gagliardo-Nirenberg and Gronwall’s inequalities over \([-1, t], t \in [-1, 0]\) lead to

\[
\sup_{t \in [-1, 0]} |(A_1^{\frac{1}{2}+\gamma}u, A_2^{\frac{1}{2}+\gamma}v)(t)|^2 
\leq |(A_1^{\frac{1}{2}+\gamma}u, A_2^{\frac{1}{2}+\gamma}v)(-1)|^2 e^{\int_{-1}^{t} \|\nabla u\|_{L^2}^2 \, ds} 
+ \int_{-1}^{t} \|\nabla u\|_{L^2}^2 \, ds \leq r_4(\omega)
\]

for some \(r_4(\omega)\) due to (34), (35) and (14). Finally, we go back to computations of (25) again to similarly obtain

\[
\frac{1}{2} \partial_t |A_1^{\frac{1}{2}+\gamma}u|^2 + |A_1^{\frac{1}{2}+\gamma}u|^2 
\leq |A_1^{\frac{1}{2}+\gamma}u|^2 + c(1 + |(A_1^{\frac{1}{2}+\gamma}u, A_2^{\frac{1}{2}+\gamma}u)|^4 + |A_1^{\frac{1}{2}+\gamma}u|^4) 
\leq |A_1^{\frac{1}{2}+\gamma}u|^2 + c(1 + \|\nabla u\|_{L^2}^2 + \|\theta\|_{L^2}^2 + |A_1^{\frac{1}{2}+\gamma}u|^2 + |A_1^{\frac{1}{2}+\gamma}u|^4).
\]

Subtracting \(|A_1^{\frac{1}{2}+\gamma}u|^2\) from both sides and applying Gronwall’s inequality give

\[
|A_1^{\frac{1}{2}+\gamma}u(0)|^2 \leq |A_1^{\frac{1}{2}+\gamma}u(-1)|^2 e^{\int_{-1}^{0} \|\nabla u\|_{L^2}^2 \, ds} 
+ \int_{-1}^{0} (1 + \|\nabla u\|_{L^2}^2 + |A_1^{\frac{1}{2}+\gamma}u|^2) \, ds.
\]

Thus, by (34), (35), (37) and (14), \(\forall \ t_0 \leq 0\), we obtain

\[
|A_1^{\frac{1}{2}+\gamma}u(0, t_0, y(t_0), \omega)|^2 \leq r_5(\omega)
\]

for some \(r_5(\omega)\). With (37) and (38) we now conclude the proof by defining \(r(\omega) \triangleq r_4(\omega) + r_5(\omega)\). □

**Proof of Theorem 2.1**

We now prove the path-wise uniqueness of the generalized solution \(y\) to (4). The issue here is that we do not know if \((u, b)(\cdot, \omega) \in L^2(t_0, T; D(A_1^{\frac{1}{2}+\gamma}) \times D(A_1^{\frac{1}{2}+\gamma}))\), in contrast to the deterministic case, due to \(z_0^\omega \in C([t_0, T]; D(A_1^{\frac{1}{2}+\gamma}))\), \(\gamma_0 > 0\) in (14), although from Proposition 3.1, we have \((\bar{u}, \bar{v})(\cdot, \omega) \in L^2(t_0, T; V_1 \times V_2)\) for \(\mathbb{P}\text{-a.e.}\ \omega \in \Omega\). The key observation from [13] is that along with the same type of classical results in [37], we only need \((u, b)(\cdot, \omega) \in L^4(t_0, T; D(A_1^{\frac{1}{2}+\gamma}) \times D(A_1^{\frac{1}{2}+\gamma}))\) to prove the path-wise uniqueness and this follows from the interpolation inequality of, e.g., in the case of \(u\),

\[
\|\nabla u\|_{L^4((t_0, T); H_1)}^4 \lesssim \|\nabla u\|_{L^\infty((t_0, T); H_1)}^2 \|\nabla u\|_{L^2((t_0, T); V_1)}^2 \lesssim 1
\]

and similarly for \(b\). Therefore, by (14) and (16), we obtain \((u, b)(\cdot, \omega) \in L^4(t_0, T; D(A_1^{\frac{1}{2}+\gamma}) \times D(A_1^{\frac{1}{2}+\gamma}))\) \(\mathbb{P}\text{-a.e.}\ \omega \in \Omega\).

Now similarly to the proof of Proposition 3.1, we may consider \((u^1, b^1, \theta^1), (u^2, b^2, \theta^2)\) that both solve (4) so that defining \(\delta u \triangleq u^1 - u^2, \delta b \triangleq b^1 - b^2, \delta \theta \triangleq \theta^1 - \theta^2\), we
obtain
\[ \begin{cases} \partial_t \delta u + A_1 \delta u + (\delta u \cdot \nabla) u^1 - (u^2 \cdot \nabla) \delta u - (\delta b \cdot \nabla) b^1 - (b^2 \cdot \nabla) \delta b = \delta \theta e_2, \\ \partial_t \delta b + A_2 \delta b + (\delta u \cdot \nabla) b^1 - (u^2 \cdot \nabla) \delta b - (\delta b \cdot \nabla) u^1 - (b^2 \cdot \nabla) \delta u = 0, \\ \partial_t \delta \theta + A_3 \delta \theta + (\delta u \cdot \nabla) \theta^1 + (u^2 \cdot \nabla) \delta \theta = \delta u_2. \end{cases} \]

Therefore, taking \( L^2 \)-inner products with \( (\delta u, \delta b, \delta \theta) \), a computation very similar to the proof of Proposition 3.1 shows that
\[
\frac{1}{2} \partial_t ||(\delta u, \delta b, \delta \theta)||_V^2 + ||(\delta u, \delta b, \delta \theta)||_V^2 \\
\leq \frac{1}{2} ||(\delta u, \delta b, \delta \theta)||_V^2 + c||((\delta u, \delta b, \delta \theta))||^4 + ||(u^1, b^1)||^4_{D(A^1) \times D(A^2)} + ||\theta^1||^2_{V_2}.
\]

Thus, Gronwall’s inequality implies \( (\delta u, \delta b, \delta \theta) = 0 \).

Next, we prove the existence of an invariant measure. With \( r(\omega) \) from (27) we let \( \Omega_N \triangleq \{ \omega \in \Omega : r(\omega) \leq N \} \) so that \( \Omega_N \not\subset \Omega \) \((N \to \infty)\). On the other hand, by Proposition 3.3
\[
\mathbb{P}(\Omega_N) < \mathbb{P}(\{ ||y(0,t_0,y(t_0),\omega)||_{D(A')} \leq N \epsilon \})
\]
where \( y(0,t_0,y(t_0),\omega) \) is a solution to (4) at time 0 with value \( y(t_0) \) at time \( t_0 \). Therefore,
\[
1 - \epsilon < \mathbb{P}(\{ ||y(0,t_0,y(t_0),\omega)||_{D(A')} \leq N \epsilon \}).
\]
By the compact embedding of \( D(A') \hookrightarrow H \) \( \forall \gamma > 0 \) (cf. Theorem 16.1 [29]), this implies that \( \forall \epsilon > 0 \), there exists a compact set \( K \subset H \) such that \( 1 - \epsilon < \mathbb{P}(y(0,t_0,y(t_0),\omega) \in K) \).

Next, we let \( \nu_T \triangleq \mathcal{L}(y(0,-\tau,y(-\tau),\omega)) \) so that \( \nu_T = P_T \delta y(0) = \mathcal{L}(y_0(\tau,\omega)) = \mathcal{L}(y_{-\tau}(0,\omega)), \tau \geq 0 \) and define
\[
\mu_T \triangleq \frac{1}{T} \int_0^T \nu_T d\tau
\]
so that we know \( \{ \mu_T \}_{T \geq 0} \) is tight. By Prokhorov’s theorem ([36]), it is weakly convergent. Thus, the existence of an invariant measure follows from the Krylov-Bogoliubov theorem ([27]), specifically Corollary 11.8 of [7].

4. Proof of Theorem 2.2: Uniqueness

We return to (4), denote by \( z^j, j = 1,2, \) the solution to (12) at \( \alpha = 0 \), but for \( t \geq 0 \) with the additional condition that \( z^j(0) = 0 \), and let \( \bar{\pi} \triangleq u - z^1, \bar{b} \triangleq b - z^2 \) and hence similarly to (17a)-(17c) we obtain
\[
\begin{cases} \partial_t \bar{\pi} + A_1 \bar{\pi} = -B_3(\bar{\pi}, \bar{\pi}) - B_3(\bar{\pi}, z^1) - B_3(z^1, \bar{\pi}) - B_3(z^1, z^1) \\
+ B_2(\bar{b}, \bar{b}) + B_2(\bar{b}, z^2) + B_2(z^2, \bar{b}) + B_2(z^2, z^2) + \theta e_2, \\
\partial_t \bar{b} + A_2 \bar{b} = -B_3(\bar{\pi}, \bar{b}) - B_3(\bar{\pi}, z^2) - B_3(z^1, \bar{b}) - B_3(z^1, z^2) \\
+ B_4(\bar{b}, \bar{\pi}) + B_4(\bar{b}, z^1) + B_4(z^2, \bar{\pi}) + B_4(z^2, z^1), \\
\partial_t \theta + A_3 \theta = -B_3(\bar{\pi}, \theta) - B_3(z^1, \theta) + \bar{\pi}_2 + z_2^1, \end{cases}
\]
where \( z_2^1 \) is the second component of \( z^1 \).
Proposition 4.1. For \( l \in \mathbb{Z}, l \geq 2 \), suppose that for both \( j = 1, 2 \), \( \sigma_j^l < \frac{C_j^l}{i^2} \). Then for all \( y_0 \in \mathbb{V}^l \), there exists a unique solution \( y = (u, b, \theta) \) to

\[
y(t) + \int_0^t A y + B(y, y) ds = y_0 + \int_0^t R y ds + G w(t)
\]

such that \( y \in C([0, T]; \mathbb{V}^l) \) \( \mathbb{P} \)-a.s., and it is a Markov process, satisfying the Feller property in \( \mathbb{V}^l \).

Proof. Due to our hypothesis that \( \sigma_j^l < \frac{C_j^l}{i^2} \), we obtain for both \( j = 1, 2 \)

\[
E[|A_j^l z_j(t)|^2] < \infty
\]

(see [12]). Moreover, due to the higher regularity of the initial data, taking \( L^2 \)-inner products of (40) with \( (A_j^l u, A_j^l b, A_j^l \theta) \) respectively and using the Banach algebra property of \( H^s, s > 1 \) in two-dimensional space, we may compute

\[
\frac{1}{2} \partial_t (|A_j^l u, A_j^l b, A_j^l \theta|^2) + L(A_j^l u, A_j^l b, A_j^l \theta)|^2
\]

\[
- \int A_j^l \partial_t (A_j^l u, A_j^l b, A_j^l \theta) + \int A_j^l \theta e_2 A_j^l u
\]

\[
= \int A_j^l (A_j^l u, A_j^l b, A_j^l \theta) + \int A_j^l \theta e_2 A_j^l u
\]

\[
\leq \frac{1}{2} (|A_j^l u, A_j^l b, A_j^l \theta|^2
\]

\[
+ c(|u|_{V^l} + |(\langle u, b, \theta \rangle + |(z, z)|_{V^l} + |(\theta, \theta)|_{V^l})^2)
\]

Subtracting \( \frac{1}{2} (|A_j^l u, A_j^l b, A_j^l \theta|^2 \) from both sides, and using the fact that we know from the basic estimate \( \mathbb{P} \)-almost surely,

\[
\sup_{t \in [0, T]} |(\langle u, b, \theta \rangle + |(z, z)|_{V^l} + |(\theta, \theta)|_{V^l})^2 dt \lesssim 1,
\]

inductively we obtain

\[
\sup_{t \in [0, T]} |(\langle u, b, \theta \rangle + |(z, z)|_{V^l} + |(\theta, \theta)|_{V^l})^2 dt \lesssim 1.
\]

With such a priori estimates, a standard Galerkin approximation scheme proves the existence of the unique solution

\[
\pi, b, \theta \in C([0, T]; V^l) \cap L^2(0, T; V^{l+1})
\]
and hence
\[ u \in C([0,T]; V^1), \quad b \in C([0,T]; V^2) \]
as \( z_j \in C([0,T]; V^1), j = 1, 2 \) by (42).

4.1. Irreducibility. Let \( B(v, \rho) \) be a ball of radius \( \rho > 0 \) centered at \( v \in V^1 \). By denseness of \( V^l \) in \( V^1 \), we find \( y_\tau \in V^{l+2} \) such that \( \|v - y_\tau\|_{V^1} < \frac{\epsilon}{2} \). Thus,
\[ P(\tau, x, B(v, \rho)) \geq P(\tau, x, B(y_\tau, \frac{\rho}{2})) \]
where \( P(t, x, \Gamma) = \mathbb{P}(\{y(t, x) \in \Gamma\}) \).

**Proposition 4.2.** For \( l \in \mathbb{Z}, l \geq 2 \), suppose that for both \( j = 1, 2, \sigma_j < \frac{C}{l^2} \). Then, given any \( \tau \in (0,T], x = (x_1, x_2, x_3) = (u_0, b_0, \theta_0) \in V^1, y_\tau \in V^{l+2} \), there exists
\[ \left\{ \begin{array}{l}
\tilde{y} \in C([0,\tau]; V^1) \cap C((0,\tau]; V^{l+2}), \\
\tilde{w}_G \in \text{Lip}([0,\tau]; V^1),
\end{array} \right. \]
such that \( \tilde{y} = (\tilde{u}, \tilde{b}, \tilde{\theta}) \) is a solution to (41) with \( \tilde{w}_G = (G^1 \tilde{w}^1, G^2 \tilde{w}^2, 0), \tilde{y}(0) = x, \tilde{y}(\tau) = y_\tau \).

**Proof.** Given \( x \in V^1 \), we consider (41) with \( G^1 \tilde{w}^1 = G^2 \tilde{w}^2 = 0 \). Then we know there exists a unique \( \tilde{y} \in C([0,T]; V^1) \cap L^2(0,T; V^{l+1}) \). Since \( \tilde{y} \in L^2(0,T; V^{l+1}) \), in particular, there exists \( \tau_1 \in (0,\tau) \) arbitrary close to 0 such that \( \tilde{y}(\tau_1) \in V^{l+1} \). Restarting from \( \tau_1 \), we find
\[ \tilde{y} \in C([\tau_1,T]; V^{l+1}) \cap L^2(\tau_1,T; V^{l+2}). \]

Now we take \( \tau_2 \in (\tau_1,\tau) \) arbitrary close to \( \tau_1 \) so that \( \tilde{y}(\tau_2) \in V^{l+2} \). We let
\[ \tilde{y}(t) = \tilde{y}(\tau_2) + \frac{t - \tau_2}{\tau - \tau_2}[y_\tau - \tilde{y}(\tau_2)] \]
for \( t \in [\tau_2,\tau] \); we observe that \( \tilde{y}(t)|_{t=\tau} = y_\tau \). We set
\[ \tilde{\xi} \triangleq \partial_t \tilde{y} + A\tilde{y} + B(\tilde{y}, \tilde{y}) - R\tilde{y}; \]

it follows that
\[ \tilde{\xi} \in V^{l+2} + C([\tau_2,\tau]; V^1) + C([\tau_2,\tau]; V^{l+1}) + C([\tau_2,\tau]; V^{l+2}). \]

Now integrating \( \tilde{\xi} \) over \([\tau_2,t]\), we see that
\[ w_G(t) \triangleq \int_{\tau_2}^{t} \tilde{\xi}(s) ds \in \text{Lip}([\tau_2,\tau]; V^1). \]

\[ \Box \]

Next, with \( \omega \in \Omega \) fixed we prove the following proposition:

**Proposition 4.3.** For \( l \in \mathbb{Z}, l \geq 2 \), suppose \( w_G, \tilde{w}_G \in C^1([0,T]; V^{l+1-\frac{2}{l^2}}), \gamma > \frac{1}{2} - \frac{2}{l^2}, \) \( x \in V^l \), for both \( j = 1, 2, \sigma_j < \frac{C}{l^2} \) and \( y, \tilde{y} \) are the corresponding solutions to (41) with \( w_G, \tilde{w}_G \) respectively. Then
\[ \|y - \tilde{y}\|_{C([0,T]; V^1)} \lesssim \|w_G - \tilde{w}_G\|_{C^1([0,T]; V^{l+1-\frac{2}{l^2}} \times V^{l+1-\frac{2}{l^2}})}. \]
Proof. By Theorem 5.2 [15], we obtain for \( j = 1, 2 \),

\[
z^j(t) = e^{-tA_j}G^jw^j(t) + \int_0^t A_j e^{-(t-s)A_j} (G^j w^j(t) - G^j w^j(s)) ds.
\]

Thus, with \( z = (z^1, z^2) \), \( \hat{z} = (\hat{z}^1, \hat{z}^2) \) where for \( j = 1, 2 \)

\[
d\hat{z}_j + A_j \hat{z}_j dt = G^j d\hat{w}_j, \quad \hat{z}_j(0) = 0,
\]

so that similarly applying Theorem 5.2 [15], we obtain for \( j = 1, 2 \)

\[
\hat{z}_j(t) = e^{-tA_j}G^j\hat{w}_j(t) + \int_0^t A_j e^{-(t-s)A_j} (G^j \hat{w}_j(t) - G^j \hat{w}_j(s)) ds
\]

and hence it follows that

\[
\|z - \hat{z}\|_{C([0,T]; V_1^1 \times V_2^1)} \lesssim \|w_G - \hat{w}_G\|_{C^\gamma([0,T]; V_1^{1+\epsilon} \times V_2^{1+\epsilon})}. \tag{47}
\]

Now by hypothesis we have \( y \) and \( \hat{y} \) as the corresponding solutions with \( w_G, \hat{w}_G \)
respectively so that defining

\[
\bar{u} \triangleq u - z^1, \quad \bar{b} \triangleq b - z^2, \quad \tilde{u} \triangleq \hat{u} - \hat{z}^1, \quad \tilde{b} \triangleq \hat{b} - \hat{z}^2
\]

and furthermore

\[
\bar{U} \triangleq \bar{u}, \quad \bar{B} \triangleq \bar{b}, \quad \Theta \triangleq \hat{\theta} - \bar{\theta}, \quad \tilde{z}^1 \triangleq \hat{z}_1 - \tilde{z}^1, \quad \tilde{z}^2 \triangleq \hat{z}_2 - \tilde{z}^2,
\]

we obtain

\[
\begin{align*}
\partial_t \bar{U} + A_1 \bar{U} + B_1(\bar{u} + \bar{z}^1, \bar{U} + \bar{z}^1) + B_1(\bar{U} + \bar{z}_1, \tilde{u} + \tilde{z}^1) &= -B_2(\bar{b} + \bar{z}^2, \bar{B} + \bar{z}^2) - B_2(\bar{B} + \bar{z}^2, \tilde{b} + \tilde{z}^2) = \Theta e_2, \\
\partial_t \bar{B} + A_2 \bar{B} + B_3(\bar{u} + \bar{z}^1, \bar{B} + \bar{z}^2) + B_3(\bar{U} + \bar{z}^1, \tilde{b} + \tilde{z}^2) &= -B_4(\bar{b} + \bar{z}^2, \bar{U} + \bar{z}^1) - B_4(\bar{B} + \bar{z}^2, \tilde{u} + \tilde{z}^1), \\
\partial_t \Theta + A_3 \Theta + B_5(\bar{u} + \bar{z}^1, \Theta) + B_5(\bar{U} + \bar{z}^1, \tilde{\theta}) &= - (\bar{U} + \bar{z}^1)_2,
\end{align*}
\]
Proposition 4.4. For $V$ by (48) and (47).

Proof. We denote by $u^i_{\alpha}$ data with Young's inequalities and Banach algebra property. Therefore, after subtracting $\frac{1}{2}(u^i_{\alpha})^2$ from both sides, by Proposition 4.1, (42) and that $\bar{U}(0) = \bar{B}(0) = \Theta(0) = 0$, we obtain

$$\|u^i_{\alpha}\|_{C([0,T],V^i)} \leq \|(\tau_1, \tau_2)\|_{C([0,T],V^i)}.$$  \hspace{1cm} (48)

Thus,

$$\|y - \hat{y}\|_{C([0,T],V^i)} \leq \|u^i_{\alpha}\|_{C([0,T],V^i)} + \|(\tau_1, \tau_2)\|_{C([0,T],V^i)} \leq \|w_G - \hat{w}_G\|_{C([0,T],V^{i+1,1-\frac{1}{2}})}.$$  \hspace{1cm} (48)

We now complete the proof that the Markovian semigroup $P_t$ is irreducible in $V^i$:

Proposition 4.4. For $l \in \mathbb{Z}, l \geq 2$, suppose that for both $j = 1, 2$, $\{\sigma_j\}_{i=1}^{\infty}$ satisfies (9), $\sigma_j < \frac{d}{r}$, $x, v \in V^i, \tau \in (0,T), \rho > 0$. Then

$$P_t(\tau, x, B(v, \rho)) > 0.$$  \hspace{1cm} (49)

Proof. We denote by $y(\cdot, w_G; x)$ the solution to (41) emanating from $x$ as initial data with $w_G = Gw = (G^1w^1, G^2w^2, 0)^T$ and obtain

$$\|y(\cdot, w_G; x) - \hat{y}(\cdot, \hat{w}_G; x)\|_{C(0,T],V^i)} \leq c_0\|w_G - \hat{w}_G\|_{W^{1,1-\frac{1}{2}}(\tau_1, V^{i+1,1-\frac{1}{2}})}$$  \hspace{1cm} (48)

by (48) and (47). \hfill \Box
for some $c_0 \geq 0$ by Proposition 4.3 if $\gamma > \frac{1}{2} - \frac{s}{n}$ and the one-dimensional Sobolev embedding of $W^{s,p}(0, \tau) \hookrightarrow C^\gamma([0, \tau]) \forall \gamma \leq s - \frac{1}{p}$. Thus,

$$\mathbb{P}(\|w_G - \hat{w}_G\|_{W^{s,p}(0,\tau;V_1^{j-1+\frac{s}{2}} \times V_2^{j-1+\frac{s}{2}})} < \frac{\rho}{2c_0}) \quad (49)$$

$$\leq \mathbb{P}(\|y(\tau, w_G; x) - \hat{y}(\tau, \hat{w}_G; x)\|_{C([0,\tau];V_1)} < \frac{\rho}{2}).$$

It is shown in [12] that

$$\mathbb{P}(\|w_G - \hat{w}_G\|_{W^{s,p}(0,\tau;V_1^{j-1+\frac{s}{2}} \times V_2^{j-1+\frac{s}{2}})} < \frac{\rho}{2c_0}) > 0 \quad (50)$$

because the law of $Gw$ is a full measure on $W^{s,p}(0,T;V_1^{j-1+\frac{s}{2}} \times V_2^{j-1+\frac{s}{2}})$. Thus, applying (50) to (49) allows us to conclude that

$$0 < \mathbb{P}(\|y(\tau, w_G; x) - \hat{y}(\tau, \hat{w}_G; x)\|_{V_1} < \frac{\rho}{2}).$$

By Proposition 4.2, we see that $0 < P(\tau, x, B(y, \frac{\rho}{2}))$. By (44), the proof is complete. \hfill \Box

4.2. Strong Feller property. We let $H_n^j = \text{span}\{g_1^j, \ldots, g_n^j\}$ where we recall that $\{g_i^j\}_{i=1}^\infty$ are the complete orthonormal systems of eigenfunctions of $A_j, j = 1, 2, 3$ and $\pi_n^j : H_j \to H_n^j, \pi_n : \mathbb{H} \to \prod_{j=1}^3 H_n^j \cong H_n$ be the projection operator defined by $\pi_n^j x = \sum_{i=1}^n \langle x, g_i^j \rangle g_i^j$. For a fixed $R > 0$, we define a cutoff function $\Psi_R \in C^\infty(\mathbb{R})$ that satisfies

$$\Psi_R(t) = \begin{cases} 1 & t \in [-R, R], \\ 0 & t \in \mathbb{R} \setminus (-R - 1, R + 1). \end{cases}$$

We consider $y_n^R(t) = (u_n^R, \theta_n^R, \phi_n^R)$, for fixed $n \in \mathbb{N}$, that solves

$$\begin{align*}
\begin{aligned}
du_n^R + & [A_1 u_n^R] + \Psi_R(\|y_n^R\|_{V_1}^2)\pi_n^1 B_1(u_n^R, u_n^R) - \Psi_R(\|y_n^R\|_{V_1}^2)\pi_n^1 B_2(b_n^R, b_n^R)]dt \\
= & \pi_n^1 \theta_n^R c_2 dt + \pi_n^1 G^1 dw^1, \\
\theta_n^R &= \pi_n^2 G^2 dw^2,
\end{aligned}
\end{align*}$$

$$\begin{align*}
\begin{aligned}
\begin{aligned}
d\theta_n^R &= [A_2 \theta_n^R] + \Psi_R(\|y_n^R\|_{V_2}^2)\pi_n^2 B_3(u_n^R, b_n^R) - \Psi_R(\|y_n^R\|_{V_2}^2)\pi_n^2 B_4(b_n^R, u_n^R)]dt \\
= & \pi_n^2 \theta_n^R c_2 dt,
\end{aligned}
\end{aligned}
\end{align*}$$

$$(u_n^R, b_n^R, \theta_n^R)(0) = (\pi_n^1 x_1, \pi_n^2 x_2, \pi_n^3 x_3) = \pi_n x,$$

where $u_n^R$ is the second component of $u_n^R$, and its associated Markovian semigroup $P_{t,n}^R$. We prove the following proposition:

**Proposition 4.5.** Under the hypothesis of Theorem 2.2, $\forall t > 0, R > 0$, there exists a constant $L = L(R, t) \geq 0$ such that

$$|P_{t,n}^R \psi(x) - F_{t,n}^R \psi(v)| \leq L\|x - v\|_V, \forall n \in \mathbb{N}, x, v \in H_n, \psi \in C_b(H_n)$$

such that $\|\psi\|_{C_b} \Delta = \sup_{x \in H_n} |\psi(x)| \leq 1$.

**Proof.** Firstly, $P_{t,n}^R$ is Feller and hence $P_{t,n}^R(C_b(V_1)) \subset C_b(V_1)$; moreover, by mean value theorem,

$$|F_{t,n}^R \psi(x) - P_{t,n}^R \psi(v)| \leq \sup_{k, h \in H_n, \|h\|_V \leq 1} |D F_{t,n}^R \psi(k) \cdot h||x - v||_V.$$
where \( DP_{t,n} \psi(k) \cdot h \) denotes the derivative of the mapping \( v \mapsto P_{t,n}^{(R)} \psi(v) \) at the point \( k \) in the direction \( h \). We rewrite (51) similarly to (4) as

\[
dy_{n}^{(R)} + [Ay_{n}^{(R)} + \Psi_{R}(||y_{n}^{(R)}||_{V'}^{2})\pi_{n}B(y_{n}^{(R)},y_{n}^{(R)})]dt = \pi_{n}Ry_{n}^{(R)}dt + \pi_{n}Gdw.
\]

Thus, by Elworthy’s formula (see e.g. pg. 267 [7]), \( \forall h \in H_{n} \) with an \( n \)-dimensional standard Wiener process \( \beta_{n} \),

\[
[DP_{t,n}^{(R)} \psi(x)] \cdot h = \frac{1}{t}E[\psi(y_{n}^{(R)}(t;x))] \int_{0}^{t} (|\pi_{n}GG^{*}\pi_{n}|^{-\frac{1}{2}}|Dy_{n}^{(R)}(s;x)| \cdot h, d\beta_{n}(s)).
\]

By Burkholder-Davis-Gundy inequality (e.g. [23]),

\[
||[DP_{t,n}^{(R)} \psi(x)] \cdot h|| \leq \frac{c}{t}||\psi||_{C_{b}}E[\left( \int_{0}^{t} (|\pi_{n}GG^{*}\pi_{n}|^{-\frac{1}{2}}|Dy_{n}^{(R)}(s;x)| \cdot h|^{2}ds \right)^{\frac{1}{2}}]
\]

where we used that \( \psi \in C_{b}(H_{n}). \) Due to (9) and (11) we have for a constant \( c \) independent of \( n \),

\[
|(|\pi_{n}GG^{*}\pi_{n}|^{-\frac{1}{2}}v)| \leq c||v||_{V'_{+}} \quad \forall \ y \in H_{n}.
\]

We let

\[
\pi_{n}^{(R)} \triangleq u_{n}^{(R)} - \pi_{n}^{1}z^{1}, \quad \tilde{\pi}_{n}^{(R)} \triangleq b_{(R)} - \pi_{n}^{2}z^{2}, \quad y_{n}^{(R)} \triangleq (\pi_{n}^{(R)}, \tilde{\pi}_{n}^{(R)}, \theta_{n}^{(R)}),
\]

where \( z^{j}, j = 1, 2 \) are solutions to (12) with \( \alpha = 0, t \geq 0, z^{j}(0) = 0 \) and also denote

\[
Y_{n}^{(R)}(t) \triangleq [Dy_{n}^{(R)}(t;x)] \cdot h
\]

where \( D \triangleq (D_{x_{1}}, D_{x_{2}}, D_{x_{3}}) \) which is a derivative of the mapping \( x \mapsto y_{n}^{(R)}(t;x) \).

Thus, \( \partial_{t}Y_{n}^{(R)} = \partial_{t}[Dy_{n}^{(R)}(t;x)] \cdot h \) and it can be checked that

\[
\left\{
\begin{aligned}
\partial_{\eta_{1}} &+ A_{1}\eta_{1} - \pi_{n}^{1}\eta_{2}, \\
2\Psi_{R}(||y_{n}^{(R)}||_{V'}^{2})(\tilde{A}^{\perp}y_{n}^{(R)} + A_{2}D_{x_{1}}y_{n}^{(R)} \cdot h)\pi_{n}^{1} &- B_{1}(u_{n}^{(R)}, u_{n}^{(R)}) + B_{2}(b_{n}^{(R)}, b_{n}^{(R)})] \\
+ \Psi_{R}(||y_{n}^{(R)}||_{V'}^{2})\pi_{n}^{1} &- B_{1}(\eta_{1}, u_{n}^{(R)}) + B_{2}(\eta_{2}, b_{n}^{(R)}) + B_{4}(b_{n}^{(R)}, u_{n}^{(R)}), \\
\partial_{\eta_{2}} &+ A_{2}\eta_{2}, \\
2\Psi_{R}(||y_{n}^{(R)}||_{V'}^{2})(\tilde{A}^{\perp}y_{n}^{(R)} + A_{2}D_{x_{1}}y_{n}^{(R)} \cdot h)\pi_{n}^{2} &- B_{2}(u_{n}^{(R)}, b_{n}^{(R)}) + B_{4}(b_{n}^{(R)}, u_{n}^{(R)}), \\
+ \Psi_{R}(||y_{n}^{(R)}||_{V'}^{2})\pi_{n}^{2} &- B_{3}(\eta_{1}, b_{n}^{(R)}) + B_{4}(\eta_{2}, u_{n}^{(R)}) + B_{4}(b_{n}^{(R)}, u_{n}^{(R)}), \\
\partial_{\eta_{3}} &+ A_{3}\eta_{3}, \\
-2\Psi_{R}(||y_{n}^{(R)}||_{V'}^{2})(\tilde{A}^{\perp}y_{n}^{(R)} + A_{2}D_{x_{1}}y_{n}^{(R)} \cdot h)\pi_{n}^{3} &- B_{5}(\eta_{1}, u_{n}^{(R)}) + B_{5}(\eta_{2}, b_{n}^{(R)}) + \pi_{n}^{3}\eta_{3},
\end{aligned}
\right\}
\]
\[
\begin{align*}
\partial_t \eta_1 + A_1 \eta_1 - \pi^1_\eta \theta_2 &= 2\Psi_R(||y^{(R)}_n||^2) \langle A^{2}_y y^{(R)}_n, A^{2}_x y^{(R)}_n \rangle \cdot h \pi^1_\eta \pi^1_\eta - B_1(u^{(R)}_n, u^{(R)}_n) + B_2(b^{(R)}_n, b^{(R)}_n) \\
&+ \Psi_R(||y^{(R)}_n||^2) \pi^1_\eta [-B_1(\eta_1, u^{(R)}_n) - B_1(u^{(R)}_n, \eta_1) + B_2(\eta_1, b^{(R)}_n) + B_2(b^{(R)}_n, \eta_1)], \\
\partial_t \eta_1 + A_2 \eta_1 &= 2\Psi_R(||y^{(R)}_n||^2) \langle A^{2}_y y^{(R)}_n, A^{2}_x y^{(R)}_n \rangle \cdot h \pi^2_\eta \pi^2_\eta - B_3(u^{(R)}_n, b^{(R)}_n) + B_4(b^{(R)}_n, u^{(R)}_n) \\
&+ \Psi_R(||y^{(R)}_n||^2) \pi^2_\eta [-B_3(\eta_2, b^{(R)}_n) - B_3(b^{(R)}_n, \eta_2) + B_4(\eta_2, u^{(R)}_n) + B_4(u^{(R)}_n, \eta_2)], \\
\partial_t \eta_2 + A_3 \eta_2 &= -2\Psi_R(||y^{(R)}_n||^2) \langle A^{2}_y y^{(R)}_n, A^{2}_x y^{(R)}_n \rangle \cdot h \pi^3_\eta B_5(u^{(R)}_n, \theta^{(R)}_n) \\
&- \Psi_R(||y^{(R)}_n||^2) \pi^3_\eta [B_5(\eta_1, \theta^{(R)}_n) + B_5(\eta_2, \theta^{(R)}_n)] + \pi^3_\eta \eta_{1,2},
\end{align*}
\]

where \( \eta_{1,2}, \eta_{2,3} \) are the second components of \( \eta_1, \eta_2, \eta_3 \) respectively.

We take \( L^2 \)-inner products with \( (A^1_1 \eta_1, A^1_2 \eta_2, A^1_3 \eta_3, A^1_4 \eta_4, A^1_5 \eta_5, A^1_6 \eta_6, A^1_7 \eta_7, A^1_8 \eta_8, A^1_9 \eta_9) \) respectively to estimate for example

\[
\frac{1}{2} \partial_t |A^{2}_1 \eta_1|^2 + |A^{2}_1 \eta_1|^2 \leq \frac{1}{2} \langle A^{2}_1 \eta_1 \rangle \langle A^{2}_1 \eta_1 \rangle + |A^{2}_1 \eta_1||A^{2}_1 \eta_1| \\
\leq \Psi_R(||y^{(R)}_n||^2) \langle A^{2}_y y^{(R)}_n, A^{2}_x y^{(R)}_n \rangle \langle A^{2}_y y^{(R)}_n, A^{2}_x y^{(R)}_n \rangle + |A^{2}_1 \eta_1||A^{2}_1 \eta_1| \\
\leq c(R) ||A^{2}_1 \eta_1||^2 + \frac{1}{2} |A^{2}_1 \eta_1|^2 + c(R) ||A^{2}_1 \eta_1||^2 \\
\leq \frac{1}{2} |A^{2}_1 \eta_1|^2 + c(R) ||A^{2}_1 \eta_1||^2,
\]

where we used Hölder’s and Young’s inequalities. Subtracting \( \frac{1}{2} |A^{2}_1 \eta_1||^2 \) from both sides give

\[
\partial_t |A^{2}_1 \eta_1|^2 + |A^{2}_1 \eta_1|^2 \leq c(R) ||A^{2}_1 \eta_1||^2.
\]

Similar computations on the other eight equations show that in sum

\[
\partial_t |A^{2}_1 \eta_4|^2 + |A^{2}_1 \eta_4|^2 \leq c(R) ||A^{2}_1 \eta_4||^2.
\]

Thus,

\[
||Y^{(R)}_n(t)||^2 \leq ||h||^2 e^{c(R)t}
\]

and hence integrating (54) gives

\[
\int_0^t ||Y^{(R)}_n(s)||^2 ds \leq c(R, t) ||h||^2 e^{c(R)t}.
\]

This implies

\[
E\int_0^t ||Y^{(R)}_n(s)||^2 ds \leq c(R, t) ||h||^2.
\]
so that
\[
E[\int_0^t \|Dg_n^{(R)}(s;x)\|_{\mathcal{V}_t}^2 \, ds] = E[\int_0^t \|g_n^{(R)}(s)\|_{\mathcal{V}_t}^2 \, ds] \leq c(R,t)\|h\|_{\mathcal{V}_t}^2. \tag{55}
\]
Therefore,
\[
\sup_{k,h \in H_n, \|h\|_{\mathcal{V}_t} \leq 1} \|Dg_n^{(R)}(s;k)\|_{\mathcal{V}_t} \leq \sup_{k,h \in H_n, \|h\|_{\mathcal{V}_t} \leq 1} \frac{c}{t} \|\psi\|_{C_b} E[\int_0^t \|\pi_n G^* \pi_n\|^{-\frac{1}{2}} |Dg_n^{(R)}(s;k)\|_{\mathcal{V}_t}^2 \, ds] \leq \sup_{k,h \in H_n, \|h\|_{\mathcal{V}_t} \leq 1} \frac{c}{t} \|\psi\|_{C_b} E[\int_0^t ||Dg_n^{(R)}(s;k)\|_{\mathcal{V}_t}^2 \, ds] \leq \frac{C(R,t)}{t}
\]
by (52), (53), Hölder’s inequality and (55).

We now consider
\[
\begin{align*}
\frac{du^{(R)}}{dt} &= [A_1 u^{(R)} + \Psi_R(\| y^{(R)} \|_{\mathcal{V}_t}^2) B_1(u^{(R)}, u^{(R)}) - \Psi_R(\| y^{(R)} \|_{\mathcal{V}_t}^2) B_2(b^{(R)}, b^{(R)})] \, dt \\
&\quad + \theta^{(R)} e^2 dt + G^1 dw^1, \\
\frac{db^{(R)}}{dt} &= [A_2 b^{(R)} + \Psi_R(\| y^{(R)} \|_{\mathcal{V}_t}^2) B_3(u^{(R)}, b^{(R)}) - \Psi_R(\| y^{(R)} \|_{\mathcal{V}_t}^2) B_4(b^{(R)}, u^{(R)})] \, dt \\
&\quad + \theta^{(R)} e^2 dt, \\
\psi(0) &= (x_1, x_2, x_3) = x,
\end{align*}
\]
with \( P_t^{(R)} \) as its associated Markovian semigroup, pass the limit and obtain the following proposition:

**Proposition 4.6.** Under the hypothesis of Theorem 2.2, \( \forall \ t > 0, R > 0 \), there exists a constant \( L = L(R,t) > 0 \) such that
\[
|P_t^{(R)} \psi(x) - P_t^{(R)} \psi(v)| \leq L \|x - v\|_{\mathcal{V}_t},
\]
\( \forall \ x, v \in \mathcal{V}_t, \psi \in C_b(\mathcal{V}_t) \) such that \( \|\psi\|_{C_b} \leq 1 \). Moreover, \( P_t^{(R)} \psi \) is Lipschitz continuous \( \forall \ \psi \in B_b(\mathcal{V}_t) \).

**Proof.** The \textit{a priori} estimates in Proposition 4.1 may be readily repeated for the Galerkin approximation system (51); hence, by the well-known compact embedding results (see [4, 28, 38]), and passing to a subsequence if necessary and relabeling it, we obtain
\[
y_n^{(R)} \to y^{(R)} \quad \text{a.e.} \ t, \ P\text{-a.s.} \ (n \to \infty).
\]
We let \( \psi \in C_b(\mathcal{V}_t) \) so that \( \psi \in C_b(H_n) \) and because \( y_n^{(R)} \in H_n \), we obtain
\[
\psi(y_n^{(R)}(\cdot,x)) \to \psi(y^{(R)}(\cdot,x)) \quad \text{in} \ L^1(0,T) \quad (n \to \infty)
\]
by dominated convergence theorem. Taking expectation, using again that \( \psi \) is continuous and bounded, due to dominated convergence theorem, we obtain
\[
E[\int_0^T |\psi(y_n^{(R)}(\cdot,x)) - \psi(y^{(R)}(\cdot,x))| \, dt] \to 0 \quad (n \to \infty)
\]
and hence by Fubini’s theorem
\[ \int_0^T E[|\psi(y_n(t,x)) - \psi(y(t,x))|]dt \to 0 \quad (n \to \infty). \]
Taking a subsequence again, and relabeling it we obtain for a.e. \( t \in [0,T], \)
\[ E[|\psi(y_n(t,x)) - \psi(y(t,x))|] \to 0 \quad (n \to \infty). \]  (56)

By (8), we now see that \( \forall x,v \in \mathbb{V}^l, \psi \in C_b(\mathbb{V}^l) \) such that \( \|\psi\|_{C_b} \leq 1 \), there exists \( L = L(R,t) \geq 0 \) such that
\[ |P_t^R(\psi(x) - P_t^R(\psi(v))| \]
\[ \leq |P_t^R(\psi(x) - P_t^R(\psi(v))| + |P_{t,n}^R(\psi(x) - P_t^R(\psi(v))| + |P_{t,n}^R(\psi(v)) - P_t^R(\psi(v))| \]
\[ \leq E[|\psi(y(\cdot;x)) - \psi(y_n(\cdot;x))|] + L\|x - v\|_{\mathbb{V}^l} + E[|\psi(y(\cdot;v)) - \psi(y_n(\cdot;v))|] \]
\[ \to L\|x - v\|_{\mathbb{V}^l}. \]
by Proposition 4.5 and (56). Since the trajectories of \( y_l(t;x) \) in \( \mathbb{V}^l \) are continuous, this holds \( \forall t \in [0,T] \). Considering the total variation of \( \mathbb{V}^l \), this holds \( \forall \psi \in B_b(\mathbb{V}^l). \)

The following proposition is now an immediate consequence:

**Proposition 4.7.** Under the hypothesis of Theorem 2.2, \( \forall t > 0, \)
\[ \lim_{R \to \infty} \|P(t,x,\cdot) - P_t^R(t,x,\cdot)\|_{\mathcal{V}_{\text{ar}}} = 0 \]
uniformly with respect to \( x \) in bounded sets of \( \mathbb{V}^l \).

**Proof of Theorem 2.2**

We now complete the proof of the strong Feller property. By Proposition 4.7 with \( x,v \) in some ball of \( \in \mathbb{V}^l, \forall \epsilon > 0 \) fixed, \( t > 0, \) there exists \( R_\epsilon > 0 \) such that \( R \geq R_\epsilon \) implies
\[ \|P(t,x,\cdot) - P_t^R(t,x,\cdot)\|_{\mathcal{V}_{\text{ar}}} + \|P(t,v,\cdot) - P_t^R(t,v,\cdot)\|_{\mathcal{V}_{\text{ar}}} < 2\epsilon. \]
Thus, for \( x,v \) such that \( \|x - v\|_{\mathbb{V}^l} < \frac{\epsilon}{L} \) where \( L \) is that of Proposition 4.6,
\[ \|P(t,x,\cdot) - P(t,v,\cdot)\|_{\mathcal{V}_{\text{ar}}} \]
\[ \leq \|P(t,x,\cdot) - P_t^R(t,x,\cdot)\|_{\mathcal{V}_{\text{ar}}} + \|P_t^R(t,x,\cdot) - P(t,x,\cdot)\|_{\mathcal{V}_{\text{ar}}} \]
\[ + \|P_t^R(t,v,\cdot) - P(t,v,\cdot)\|_{\mathcal{V}_{\text{ar}}} \leq 2\epsilon + L\|x - v\|_{\mathbb{V}^l} \leq 3\epsilon \]
by Proposition 4.6. Thus, \( P_t : B_b(\mathbb{V}^l) \mapsto C_b(\mathbb{V}^l) \) \( \forall t \in (0,T] \); i.e. \( \{P_t\} \) is strong Feller.

**5. Appendix**

**Lemma 5.1.** ([10]) Let \( \{g_i^j, \lambda_i^j\}_{i=1}^\infty \) be the eigenvectors and their corresponding eigenvalues of \( A_j, j = 1,2 \). Then

1. for \( f = \sum_{i=1}^\infty f_i(t)g_i^j(x), j = 1,2, \) \( \mathbb{P}\)-a.s.,
\[ f \in D(A_j^k) \text{ if and only if } \sum_{i=1}^\infty (f_i(t))^2(\lambda_i^j)^{2k} < \infty; \]
(2) in particular, for \( G^j w^j(t) = \sum_{i=1}^{\infty} \sigma_i^j \beta_i(t) g_i^j, j = 1, 2 \), from (5) satisfies \( \mathbb{P} \)-a.s.,
\[
G^j w^j(t) \in D(A_k^j) \text{ if and only if } \sum_{i=1}^{\infty} (\sigma_i^j)^2(\lambda_i^j)^{2k} < \infty.
\]

**Lemma 5.2.** (Proposition 6.1 [10]; see also Chapter III, Section 3.5.2 [38])
For \( z_i^j(t) = \int_{-\infty}^{\infty} e^{(-A_j^i-\alpha)(t-s)} dG^j w^j(s) \) in (13), where \( G^j w^j(s) = \sum_{i=1}^{\infty} \sigma_i^j \beta_i(s) g_i^j, j = 1, 2 \) from (5),

(1) \( \mathbb{P} \)-a.s.,
\[
z_i^j \in D(A_k^j) \text{ if and only if } \sum_{i=1}^{\infty} \frac{(\sigma_i^j)^2(\lambda_i^j)^{2k}}{2(\lambda_i^j + \alpha)} < \infty;
\]

(2) \[
\lim_{\alpha \to \infty} \mathbb{E}[|A_k^j z_i^j|^4] = 0.
\]

**Lemma 5.3.** (Lemma 2.2 [21]) In an \( n \)-dimensional bounded domain \( D \), let \( P \) be a continuous projector from \( (L^\alpha(D))^n \) to the closure of \( \{ u \in (C^\infty(D))^n : \nabla \cdot u = 0 \} \) in \( (L^\alpha(D))^n \) and \( A = -P \Delta \). Then for \( 0 \leq \delta < \frac{1}{2} + \frac{n(1-\frac{1}{2})}{2} \)
\[
\| A^{-\delta} P(u \cdot \nabla)v \|_{L^\nu} \lesssim_{\delta, \theta, \rho} \| A^\theta u \|_{L^\nu} \| A^\rho v \|_{L^\nu}
\]
if \( \delta + \theta + \rho \geq \frac{2}{\nu} + \frac{1}{2}, \theta > 0, \rho > 0, \rho + \delta > \frac{1}{2} \).

**Proposition 5.4.** (Proposition 6.1 [10]; see also Chapter III, Section 3.5.2 [38])
Suppose \(-1 \leq \theta(x, t_0) \leq 1 \) a.e. \( x \in D \). Then \(-1 \leq \theta(x, t) \leq 1 \) a.e. \( x \in D, \forall t \geq t_0 \), \( \mathbb{P} \)-almost surely.

**References**


