Well-posedness of Hall-magnetohydrodynamics system forced by \textit{Lévy} noise

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ABSTRACT. We establish the existence and uniqueness of a local smooth solution to the Cauchy problem for the Hall-magnetohydrodynamics system that is inviscid, resistive, and forced by multiplicative \textit{Lévy} noise in the three dimensional space. Moreover, when the initial data is sufficiently small, we prove that the solution exists globally in time in probability; that is, the probability of the global existence of a unique smooth solution may be arbitrarily close to one given the initial data of which its expectation in a certain Sobolev norm is sufficiently small. The proofs go through for the two and a half dimensional case as well. To the best of the authors’ knowledge, an analogous result is absent in the deterministic case due to the lack of viscous diffusion, exhibiting the regularizing property of the noise. Our result may also be considered as a physically meaningful special case of the extension of work of \cite{27, 37} from the first-order quasilinear to the second-order quasilinear system because the Hall term elevates the Hall-magnetohydrodynamics system to the quasilinear class, in contrast to the Naiver-Stokes equations that has most often been studied and is semilinear.

Keywords: Hall-magnetohydrodynamics system; \textit{Lévy} noise; well-posedness.

1. INTRODUCTION

The magnetohydrodynamics (MHD) system consists of a coupling of the Maxwell’s equation and Navier-Stokes equations (NSE) of fluid mechanics. The pioneering works of Batchelor \cite{4} and Chandrasekhar \cite{15} on the magnetic properties of electrically conducting fluids that is expressed through the MHD system have attracted immediate and extensive interests from researchers in various disciplines such as physics, mathematics and engineering. On the other hand, the study of the Hall-MHD system may be traced back as far as that of Lighthill \cite{29} Section 8 and Section 9 (see also \cite{6} Section 3): it consists of the MHD system with an addition of the Hall term that actually arises upon writing current density as the sum of the ohmic current and a Hall current that is perpendicular to the magnetic field (see \cite{29} (94)). The Hall effect is considered to be of fundamental importance in astrophysical plasma as it modifies small scale turbulent activity, producing departure from MHD predictions. The analysis of \cite{17} also revealed its important role in turbulence, in which magnetic islands simultaneously reconnect in a complex manner (see also \cite{22}). The study of other effects of the Hall term has also been pursued; e.g. star formation by Wardle \cite{46}, and the enhancement of the energy exchange

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by Miura and Hori [35] (see also [36]). Despite such importance of the Hall term in physics and engineering, mathematically it heightens the level of nonlinearity of the viscous resistive MHD system from a second-order semilinear to a second-order quasilinear level, significantly making its qualitative analysis more difficult. Consequently, as we will elaborate shortly, it is only in the recent years, more than half a century since the work of Lighthill [29], that the mathematical results have started to flourish taking advantage of the special structure of the Hall term.

On the other hand, the analysis of equations in fluid mechanics forced by random noise has a long history since the work of [5] but concentrated mostly on the NSE forced by noise that is white in time. Even the MHD system forced by random noise has received relatively little attention and only recently due to its complexity of the coupling with the Maxwell’s equation (e.g. [3, 16, 33, 40, 43, 44, 49]). For the Hall-MHD system forced by noise that is white-in-time, the author in [51] proved the global existence of a martingale solution in three-dimensional (3−d), as well as two-and-a-half-dimensional (212−d) torus. The purpose of this work is to introduce the inviscid resistive Hall-MHD system forced by Lévy noise, and in particular prove its local well-posedness (see Theorem 2.1), as well as the global well-posedness in probability in case the initial data is sufficiently small (see Theorem 2.3). To the best of the authors’ knowledge, an analogous result does not exist in the deterministic case, demonstrating the regularizing effect of the noise. As we will elaborate, our proof is strongly inspired by the work of Kim [27] and Mohan and Sritharan [37] (also [8]) and may be considered as its extension from the first-order quasilinear system to a physically meaningful special case of a second-order quasilinear system. However, there are significant differences in comparison to their proofs. Firstly, the proof within the work of Kim in [27], which focused on a noise in the form of a cylindrical Wiener process, cannot go through in the case of the Lévy noise. Secondly, the work of Mohan and Sritharan in [37] uses Minty-Browder technique while we use the truncation and Cauchy property. Finally, as we will see, our estimates are very much specifically adapted to the Hall term; it is not clear if the computations within [27, 37] will go through for the Hall term.

2. Statement of main results

Let us fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})\) with \(\mathbb{E}\) the expectation with respect to \(\mathbb{P}\), \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), \(n = 2\) or \(3\). We denote by \(u(t, x) \triangleq (u_1, u_2, u_3)(t, x)\), \(b(t, x) \triangleq (b_1, b_2, b_3)(t, x)\), \(j \triangleq \nabla \times b\) and \(\pi(t, x)\), the velocity, magnetic, current density and pressure fields, respectively. Let us also represent the viscosity which is the reciprocal of the kinetic Reynolds number, resistivity which is the reciprocal of the magnetic Reynolds number, and the Hall parameter by \(\nu, \eta, \epsilon \geq 0\), respectively. Finally, we denote the slowly and rapidly fluctuating forcing terms by mappings \(g_1, g_2\) and mappings \(f_1, f_2\), respectively. We may now write the Hall-MHD system of our main concern as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi &= j \times b + \nu \Delta u + g_1 + f_1, \\
\frac{\partial b}{\partial t} - \nabla \times [(u - \epsilon j) \times b] &= \eta \Delta b + g_2 + f_2, \\
\nabla \cdot u &= \nabla \cdot b = 0, \quad (u_0, b_0)(x) \triangleq (u, b)(x, 0).
\end{align*}
\]
We immediately note that letting $\epsilon = 0$, the system recovers the MHD system and furthermore letting $b \equiv 0$ deduces the NSE. For brevity, hereafter we write $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ for $i \in \{1, 2, 3\}$, $\int f = \int_{\mathbb{R}^3} f dx$, $X \triangleq (u, b)$, $X_0 \triangleq (u_0, b_0)$, $A \approx_{a, b} B$, $A \lesssim_{a, b} B$ to indicate the existence of a non-negative constant $C = C(a, b)$ such that $A = CB$, $A \leq CB$ respectively. Moreover, for simplicity of arguments let us assume that $\epsilon = 1, g_1 \equiv g_2 \equiv 0$; our results in the general case when $\epsilon \neq 1$ may be retained with more careful computations while the case when $g_1, g_2$ do not necessarily vanish is also a matter of imposing appropriate hypothesis and additional standard computations (see e.g. [51]). Finally, throughout the rest of the manuscript, we shall consider $f_1, f_2$ to be multiplicative noise; the case in which they are additive may be analogously studied (see [36]).

Let us briefly review some relevant results of mathematical analysis on the Hall-MHD system. Due to the complexity of the Hall term, it was as recent as 2011 when Acheritogaray et al. [2] finally proved the global existence of a weak solution to the deterministic Hall-MHD system in $T^3$. In [9], the authors considered the 3−d deterministic Hall-MHD system with $x \in \mathbb{R}^3$, and in particular proved the global existence of a weak solution if $\nu > 0$ and $\eta > 0$, the local well-posedness and a blow-up criterion with $X_0 \in H^m(\mathbb{R}^3)$ for $m > \frac{3}{2}$ if $\nu \geq 0, \eta > 0$, as well as the global well-posedness for sufficiently small initial data if $\nu > 0$ and $\eta > 0$.

We emphasize in particular that the global well-posedness for sufficiently small initial data required both $\nu > 0$ and $\eta > 0$. Some temporal decay estimates for weak solutions to the deterministic Hall-MHD system are established in [11]. The results of a blow-up criterion and a global well-posedness with small initial data from [9] are improved in [10] (see also [14]): however, their results of the global well-posedness for sufficiently small initial data also required both $\nu > 0$ and $\eta > 0$.

On the other hand, Chae and Weng in [13] proved that the deterministic viscous non-resistive Hall-MHD system is not globally well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{3}{2}$. In case the Hall-MHD system is forced randomly, the author in [51] considered the noise that is white-in-time and proved the global existence of a martingale solution in both 3−d and $2\frac{1}{2}−d$ cases; we also refer to [18] for some study of stochastic line motion and flux conservation for nonideal MHD-type systems including the Hall-MHD system.

The difficulty of mathematical analysis on the Hall term is most certainly worth an emphasis. Again, we emphasize that even the global existence of a weak solution, which is perhaps the very first step toward a complete theory of well-posedness, was established by Acheritogaray et al. [2] as recently as in 2011. Let us describe two very fundamental results which are well known for the NSE and the MHD system, and yet remain completely open for the Hall-MHD system. Firstly, since as far back as the works of Leray [28] in 1934 and Sermange and Temam [41] in 1978, the uniqueness of the global weak solution to the NSE and the MHD system with $\nu > 0, \eta > 0$ are both known in case $x \in \mathbb{R}^2$. Remarkably this is an open and very difficult problem for the case of the Hall-MHD system. Secondly, the proof of the local well-posedness of the NSE and the MHD system even with $\nu = \eta = 0$ and in any dimension may be found in various sources in the literature (e.g. [31]). In striking contrast, this is also an open problem in the case of the Hall-MHD system. The best result in the literature is that by Chae, Wan and Wu [12] in which the authors proved the local well-posedness of the Hall-MHD system with zero viscous
diffusion and positive magnetic diffusion in the form of $\eta(-\Delta)^{\frac{3}{2}}$ with $\beta > 1$ instead of $-\eta\Delta$ where $(-\Delta)^{\frac{3}{2}}$ is a Fourier operator with a Fourier symbol of $m(\xi) = |\xi|^\beta$.

In preparation to state our main results, let us denote by $(\cdot, \cdot)$ and $|\cdot|$ the $L^2(\mathbb{R}^n)$-inner product and $L^2(\mathbb{R}^n)$-norm respectively. Moreover, let us denote by $\| \cdot \|_{H^s}$ the norm of Sobolev space $H^s(\mathbb{R}^n)$, while $(\cdot, \cdot)$ the duality pairing and finally $\langle \cdot, \cdot \rangle$ the quadratic variation. Additionally, let us define $\mathcal{H} \equiv \{ f \in L^2(\mathbb{R}^n) : \nabla \cdot f = 0 \}$, and $P_H : L^2(\mathbb{R}^n) \mapsto \mathcal{H}$ the Helmholtz-Leray projection, which is known to commute with $\Delta$ on $\mathbb{R}^n$ as can be seen immediately by considering their Fourier symbols. We also write

$$\frac{\partial}{\partial t} A_1 \equiv -\nu \Delta, \quad \frac{\partial}{\partial t} A_2 \equiv -\eta \Delta, \quad A \Psi \equiv \left( \frac{\partial}{\partial t} A_1 \phi \right) \forall \Psi = (\phi, \psi).$$

We also define

$$B_1(\phi^1, \phi^2, \phi^3) \equiv \langle \rho_H(\phi^1 \cdot \nabla)(\phi^2, \phi^3) \rangle,$$

$$B_2(\phi^1, \phi^2, \phi^3) \equiv \langle \rho_H(\phi^1 \cdot \nabla)(\phi^2, \phi^3) \rangle,$$

$$B_3(\phi^1, \phi^2, \phi^3) \equiv \langle \rho_H(\phi^1 \cdot \nabla)(\phi^2, \phi^3) \rangle,$$

$$B_4(\phi^1, \phi^2, \phi^3) \equiv \langle \rho_H(\phi^1 \cdot \nabla)(\phi^2, \phi^3) \rangle,$$

and

$$\langle B(\Psi_1, \Psi_2) \rangle \equiv \langle B_1(\phi^1, \phi^2, \phi^3) \rangle - \langle B_2(\psi^1, \psi^2, \psi^3) \rangle + \langle B_3(\phi^1, \psi^2, \psi^3) \rangle - \langle B_4(\psi^1, \phi^2, \phi^3) \rangle \quad (5)$$

for $\Psi_i = (\phi^i, \psi^i)$ where $i \in \{1, 2, 3\}$, and we also write e.g. $B_1(\phi^1, \phi^1) \equiv B_1(\phi^1)$, $B(\Psi_1, \Psi_1) \equiv B(\Psi_1)$. Finally, let us also denote for the Hall term,

$$(H \Psi_1, \Psi_2) \equiv \langle \rho_H(\phi^1, \phi^1) \rangle \equiv \rho_H(\phi^1, \phi^1).$$

We also write $J^s \equiv (1 - \Delta)^{\frac{s}{2}}$ for $s \in \mathbb{R}$, $B(\mathcal{H} \setminus \{0\})$ the Borel $\sigma$-field generated by $\mathcal{H} \setminus \{0\}$ and $Z$ a measurable space where the solution has its paths such that $Z \in B(\mathcal{H} \setminus \{0\})$. To specify the forcing terms, we denote by $\Phi_i, i \in \{1, 2\}$, the operators such that $\Phi_i : \mathcal{H} \to L^2(\mathcal{H}, H^s(\mathbb{R}^n))$ and $\text{Tr}_{\mathcal{H}}((J^s \Phi_i)^* J^s \Phi_i) < \infty$ for all $s \geq 0$ where $L^2(\mathcal{H}, H^s(\mathbb{R}^n))$ is the space of all Hilbert-Schmidt operators from $\mathcal{H}$ to $H^s(\mathbb{R}^n)$. We also let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis in $\mathcal{H}$, and $\{\beta_i\}_{i=1}^\infty$ for $i \in \{1, 2\}$, sequences of one-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ so that $W_i \equiv \sum_{j=1}^\infty e_j \beta_i^j$ represents a cylindrical Wiener process defined on $\mathcal{H}$. For the purpose of subsequent computations, we observe that

$$\text{Tr}((\Phi_i(f))^* \Phi_i(f)) = \sum_{j=1}^\infty ((\Phi_i(f))^* \Phi_i(f))e_j e_j = \|\Phi_i(f)\|^2_{L^2(\mathcal{H}, \mathcal{H})}. \quad (7)$$

Let us furthermore denote $\lambda(dz)$ to be a $\sigma$-finite Lévy measure on $(Z, B(Z))$ with an associated Poisson random measure $\mathcal{N}(dt, dz)$ and $\tilde{\mathcal{N}}(dt, dz) \equiv \mathcal{N}(dt, dz) - \lambda(dz)dt$ represent the compensated Poisson random measure so that $\mathbb{E}[\mathcal{N}(dt, dz)] = \lambda(dz)dt$ (see e.g. [11], pg. 105). Let us also denote jump coefficients $\gamma_i : H^s(\mathbb{R}^n) \times Z \mapsto H^s(\mathbb{R}^n)$. With such, we define

$$\Phi(X) dW(t) \equiv \left( \Phi_1(X) dW_1(t) \right), \quad \gamma(X) \tilde{\mathcal{N}}(dt, dz) \equiv \left( \gamma_1(X, z) \tilde{N}_1(dt, dz) \right), \quad \gamma_2(X, z) \tilde{N}_2(dt, dz). \quad (8)$$
and choose $P_H f_i = \Phi_i(X(t))dW_i + \int_Z \gamma_i(u(t^-), b(t^-), z)\tilde{N}_i(dt, dz)$ for $i \in \{1, 2\}$. We apply $P_H$ to (1a) and (1b) and use identities (16b) and (16a) to deduce
\[
du = [-P_H[(u \cdot \nabla)u] + P_H[(b \cdot \nabla)b] + \nu\Delta u]dt
+ \Phi_1(X(t))dt + \int_Z \gamma_1(X(t^-), z)\tilde{N}_1(dt, dz),
\]
\[
db = [-P_H[(u \cdot \nabla)b] + P_H[(b \cdot \nabla)u] - P_H[\nabla \times (j \times b)] + \eta\Delta b]dt
+ \Phi_2(X(t))dt + \int_Z \gamma_2(X(t^-), z)\tilde{N}_2(dt, dz),
\]
where $X(t^-)$ denotes the left limit of $X$ at $t$. We are now ready to state the definition of a local solution; hereafter we focus on the dimension $n = 3$; in Remark 2.1 we elaborate on the other case.

**Definition 2.1.** Given $X_0 = (u_0, b_0) \in L^{p_0}(\Omega; H^s(\mathbb{R}^3)) \cap \mathcal{H}$ where $p_0 \in [4, \infty), s > \frac{7}{2}$, we define a pair $(X, \tau)$ where $X = (u, b)$ to be a local strong (path-wise unique) solution to the Hall-MHD system (1a)-(1c) if
(1) $\tau$ is a strictly positive stopping time; i.e. $\mathbb{P}(\{\omega \in \Omega : \tau(\omega) > 0\}) = 1$, and $\tau(\omega) \triangleq \lim_{N \to \infty} \tau_N(\omega)$ for almost all $\omega \in \Omega$ where $\tau_N(\omega) \triangleq \inf\{t \geq 0 : \|X(t, \omega)\|_{H^r} \geq N\}$ with it being $+\infty$ in case the infimum is not attained,
(2) for all $t > 0$, $X$ is a progressively measurable stochastic process such that $X \in L^{p_0}(\Omega; L^\infty([0, T]; H^s(\mathbb{R}^3)))$ and the paths of $X$ belong to $D(0, t; H^s(\mathbb{R}^3))$ where $X(\cdot) = (u, b)(\cdot \wedge \tau)$ and $D(0, t; H^s(\mathbb{R}^3))$ is the space of all càdlàg paths from $[0, t]$ to $H^s(\mathbb{R}^3)$,
(3) $X$ satisfies
\[
X(t \wedge \tau) = X_0 + \int_0^{t \wedge \tau} [-AX(\delta) - B(X(\delta)) - HX(\delta)]d\delta
+ \int_0^{t \wedge \tau} \Phi(X(\delta))dW(\delta) + \int_0^{t \wedge \tau} \int_Z \gamma(X(\delta^-), z)\tilde{N}(d\delta, dz).
\]

We now state our main results.

**Theorem 2.1.** Let $\nu = 0$ and $\eta > 0$ in (1a) and (1b). For all $\delta > 0$, suppose $\Phi_i(\cdot) : H^{\delta}(\mathbb{R}^3) \rightarrow L^2(\mathcal{H}, H^{\delta}(\mathbb{R}^3))$ and $\gamma_i(\cdot, \cdot) : H^{\delta}(\mathbb{R}^3) \times Z \rightarrow H^{\delta}(\mathbb{R}^3), i \in \{1, 2\}$, satisfy the following conditions:
(1) *(Growth condition)* There exists a constant $K > 0$ such that for all $p \in \mathbb{N}$, $Y \in H^p(\mathbb{R}^3)$ and $t \geq 0$,
\[
\|\Phi(Y)\|^2_{L^2(\mathcal{H}, H^\delta)} + \int_Z \|\gamma(Y, z)\|^2_{H^\delta} \lambda(dz) \leq K(1 + \|Y\|^2_{H^\delta}).
\]
(2) *(Lipschitz condition)* There exists a constant $L > 0$ such that for all $t \geq 0$ and $Y, Y^2 \in H^\delta(\mathbb{R}^3),$
\[
\|\Phi(Y^1) - \Phi(Y^2)\|^2_{L^2(\mathcal{H}, H^\delta)} + \int_Z \|\gamma(Y^1, z) - \gamma(Y^2, z)\|^2_{H^\delta} \lambda(dz) \leq L\|Y^1 - Y^2\|^2_{H^\delta}.
\]
Then for all $X_0 \in L^{p_0}(\Omega; H^s(\mathbb{R}^3) \cap \mathcal{H})$ where $p_0 \in [4, \infty), s > \frac{7}{2}$, there exists a path-wise unique local strong solution $(X, \tau)$ to the Hall-MHD system (1a)-(1c).

**Theorem 2.2.** Suppose that the hypothesis of Theorem 2.1 holds. Then for any $\delta \in (0, 1)$ and the local path-wise unique strong solution $(X, \tau)$ to the Hall-MHD system (1a)-(1c) corresponding to an initial data $X_0 \in L^{p_0}(\Omega; H^s(\mathbb{R}^3)) \cap \mathcal{H}$ where
for a similar study on the fluid equations in which the equation of the velocity is inviscid while the other equation is diffusive, we refer to [50] for the 2-d Hall-MHD system (e.g. [14], [22] as well as [31], Chapter 2.3.1) for the 2-d Hall-MHD system, let us denote the horizontal components of the solution by $u_h \triangleq (u_1, u_2), b_h \triangleq (b_1, b_2)$ so that consequently the horizontal current density is $j_h \triangleq (\partial_2 b_3, -\partial_1 b_3)$, as well as a horizontal gradient and a horizontal Laplacian by $\nabla_h \triangleq (b_1, b_2), \Delta_h \triangleq \sum_{i=1}^2 \partial_i \partial_i$, respectively. Under these notations, making use of (16a)-(16c), we may deduce the following 2-d version of the system (1a)-(1c)

$$
\partial_t u + (u_h \cdot \nabla_h) u + \nabla_h \Pi = (b_h \cdot \nabla_h)b + \nu \Delta_h u + f_1,
$$

$$
\partial_t b + (u_h \cdot \nabla_h)b + \nabla_h \times (j_h \times b) = \eta \Delta_h b + (b_h \cdot \nabla_h)u + f_2,
$$

where $\Pi \triangleq \pi + |b_h|^2$ denotes the total pressure. Throughout the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3, it is clear that the computations may be modified so that analogues of our results will continue to hold in the 2-d case; considering the notation issues that will arise, we chose to focus on the 3-d case in this manuscript.

Remark 2.2. For a similar study on the fluid equations in which the equation of the velocity is inviscid while the other equation is diffusive, we refer to [50] in which Boussinesq system with partial viscous diffusion or partial thermal diffusion is studied. Theorem 2.1, Theorem 2.2 and Theorem 2.3 may be proven for the Boussinesq system, even in the case of zero viscous and thermal diffusion.

Remark 2.3. We remark again that Chae and Weng in [13] proved that the deterministic viscous non-resistive Hall-MHD system is not globally well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. In contrast, Theorem 2.3 demonstrates a type of regularizing property of the noise that leads to the global well-posedness of the inviscid resistive Hall-MHD system in probability.
We look forward to future work in which we shall try to extend Theorem 2.1, Theorem 2.2 and Theorem 2.3 to the inviscid non-resistive Hall-MHD system; the singularity of the Hall term seems to create major difficulty. Moreover, investigating whether the regularizing effect of the noise may bring about a resolution to the path-wise uniqueness of the 2 1/2-d Hall-MHD system is also a challenging open problem of much interest.

Remark 2.4. The global well-posedness of the 2-d inviscid resistive deterministic MHD system has caught much attention very recently (e.g. [7, 19, 23, 24, 45, 47, 48, 53, 54]) and still remains an outstanding open problem. Even with initial data sufficiently small, its global well-posedness seems very difficult to prove due to the lack of viscous diffusion. On the other hand, it is clear that the proofs of Theorem 2.1 and Theorem 2.3 may be extended to the case \( \epsilon = 0 \) and \( \eta = 0 \) in the 2-d case, establishing such a result in case the system is forced by Lévy noise, demonstrating its regularizing effect.

Remark 2.5. The precise smallness of \( K_3 = K_3(K_1, K_2, \lambda(Z)) > 0 \) will be specified in the proof of Theorem 2.3; it also depends on general constants from the Sobolev embedding of \( H^s(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3) \) and the commutator estimate Lemma 3.1. In relevance, we remark that the authors in [35] proved an analogous result to Theorem 2.3 for the NSE with hereditary viscosity under a slight variation of the conditions (14a)-(14c); mathematically, neither the condition therein or (14a)-(14c) implies each other.

In the following sections, we first state some preliminaries and thereafter prove Theorem 2.1, Theorem 2.2 and Theorem 2.3. The proof of the well-posedness consists in particular of the stochastic analogue of the Fourier truncation method introduced by the authors in [20]. Throughout the technical computations, we focus our attention mainly to the Hall term, which create significant difficulty in comparison to other systems such as the MHD system. For example, although the authors in [33] proved the global well-posedness of the 2-d MHD system perturbed by Lévy noise, it is not clear to the us if the monotonicity argument used therein is applicable for the system (1a)-(1c) due to the Hall term (see the discussion before Proposition 4.1 concerning the difficulty the Hall term represents).

3. Preliminaries

As we have seen already in the formulation of (9), it is useful for us to recall the following vector calculus identities: for any \( \Theta, \Psi \in \mathbb{R}^3 \),

\[
\nabla \times (\Theta \times \Psi) = \Theta(\nabla \cdot \Psi) - \Psi(\nabla \cdot \Theta) + (\Psi \cdot \nabla)\Theta - (\Theta \cdot \nabla)\Psi, \tag{16a}
\]

\[
(\nabla \times \Theta) \times \Theta = -\nabla \left( \frac{|\Theta|^2}{2} \right) + (\Theta \cdot \nabla)\Theta, \tag{16b}
\]

\[
(\Theta \times \Psi) \cdot \Theta = 0. \tag{16c}
\]

Let us also recall

Lemma 3.1. (26) Let \( f, g \) be smooth functions which satisfy

\[
\nabla f \in L^p(\mathbb{R}^3), \quad \Lambda s^{-1}g \in L^{p_2}(\mathbb{R}^3), \quad \Lambda s^f \in L^{p_3}(\mathbb{R}^3), \quad g \in L^{p_4}(\mathbb{R}^3),
\]

where \( p \in (1, \infty), \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \) \( p_2, p_3 \in (1, \infty), s > 0 \) and \( \Lambda \delta = (-\Delta)^{\delta} \) is defined through Fourier transform. Then there exists a constant \( c > 0 \) that is
independent of \( f \) and \( g \) such that
\[
\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq c(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).
\] (17)

Let us also define the Fourier truncation \( S_R \) defined by \( \tilde{S}_R f(\xi) = 1_{B_R}(\xi) f(\xi) \) where \( B_R \) denotes the ball of radius \( R \) centered at the origin, and state its properties:
\[
\|S_R f\|_{H^s} \leq \|f\|_{H^s}, \quad (18a)
\]
\[
\|S_R f - f\|_{H^s} \lesssim \left( \frac{1}{R} \right)^k \|f\|_{H^{s+k}}, \quad (18b)
\]
\[
\|(S_R - S_{R'}) f\|_{H^s} \lesssim \max \left\{ \left( \frac{1}{R} \right)^k, \left( \frac{1}{R'} \right)^k \right\} \|f\|_{H^{s+k}}, \quad (18c)
\]
for \( k \geq 0 \). The proofs of these properties may be found on [20, pg. 1042]. We also recall the following Gronwall’s inequality type result:

**Lemma 3.2.** ([10] Lemma A.1) Let \( E(t), Y(t), V(t) \) and \( \phi(t) \) be non-negative processes and \( Z(t) \) a non-negative integrable process. Suppose that \( V(t) \) is non-decreasing in \( t \) and there exist non-negative constants \( C, \alpha, \beta, \gamma, \delta \) such that
\[
\int_0^T \phi(r) dr \leq C \quad \mathbb{P}\text{-a.s.,} \quad 2\beta e^C \leq 1, \quad 2\delta e^C \leq \alpha, \quad (19a)
\]
\[
E(t) + \alpha Y(t) \leq Z(t) + \int_0^t \phi(r) E(r) dr + V(t), \quad \mathbb{P}\text{-a.s., for } 0 \leq t \leq T, \quad (19b)
\]
\[
\mathbb{E}[V(t)] \leq \beta \mathbb{E}[E(t)] + \gamma \int_0^t \mathbb{E}[E(r)] dr + \delta \mathbb{E}[Y(t)] + \tilde{C}, \quad \text{for } 0 \leq t \leq T. \quad (19c)
\]
where \( \tilde{C} > 0 \) is a constant. If \( E \in L^\infty([0,T] \times \Omega) \), then
\[
\mathbb{E}[E(t) + \alpha Y(t)] \leq 2 e^{C + 2\gamma e^C} (\mathbb{E}[Z(t)] + \tilde{C}), \quad t \in [0, T].
\]

Finally, let us hereafter keep in mind that an infimum over \( \mathbb{R}_+ \) of an empty set will always be \( +\infty \); this will simply our notations upon introducing various stopping times.

4. **Proof of Theorem** [21]

For any \( N > 0 \), let us define a Lipschitz continuous function \( \psi_N : [0, \infty) \rightarrow [0, 1] \) by
\[
\psi_N(y) \triangleq \begin{cases} 
1 & \text{if } y \in [0, N], \\
N + 1 - y & \text{if } y \in (N, N + 1], \\
0 & \text{if } y \in (N + 1, \infty),
\end{cases} \quad (20)
\]
with which we consider the following Hall-MHD system with the cutoff:
\[
dX(t) = -AX(t)dt + \psi_N(k) [-B(X(t)) - HX(t)]dt \\
+ \Phi(X(t))dW(t) + \int_Z \gamma(X(t-), z)\tilde{N}(dt, dz), \quad (21)
\]
where \( k \triangleq \|X\|_{H^s} \) for \( \ell \in \left( \frac{s}{2}, s-1 \right) \). We consider the truncated initial data \( X_R(0) \triangleq S_R X_0 \) so that solution \( X^R = (u^R, b^R) \) to the following truncated Hall-MHD system.
with the cutoff
\[ dX^R(t) = -AX^R(t)dt + \psi_N(k^R)S_R[-B(X^R(t)) - HX^R(t)]dt \quad (22a) \]
\[ + SR\Phi(X^R(t))dW + \int_Z S_R\gamma(X^R(t-), z)\tilde{N}(dt, dz), \]
\[ X^R(0) = S_RX_0, \quad (22b) \]
where \( k^R \triangleq \|X^R\|_{H^\lambda} \), for \( \lambda \in (\frac{5}{2}, s - 1) \), will continue to lie in the space \( \mathcal{H}_R \triangleq \mathcal{S}_R\mathcal{H} \); we note that we used the fact that \( \mathcal{S}_RAX^R = A\mathcal{S}_RX^R = AX^R \).

It is immediate that the solution to \((22a), (22b)\) exists on time interval \([0, T)\) where \( T = T(N) > 0 \). Indeed, e.g. we may estimate the nonlinear term
\[ \|\psi_N(k^R)S_R\nabla(u^R \cdot \nabla)u^R\|_{L^2} \lesssim \psi_N(k^R)\|u^R\|_{L^\infty}\|\nabla u^R\|_{L^2} \]
\[ \lesssim \psi_N(k^R)\|u^R\|_{L^2}^\frac{2}{\lambda+2}\|u^R\|_{H^\lambda}^\frac{\lambda}{\lambda+2}\|u^R\|_{L^\infty}\|u^R\|_{H^\lambda}^\frac{\lambda}{\lambda+2} \lesssim c(N)(1 + \|u^R\|_{L^2}^2) \]
by Hölder’s inequality, Gagliardo-Nirenberg’s inequalities and \((20)\). The other three nonlinear terms may be estimated analogously. For the Hall term, we may estimate
\[ \|\psi_N(k^R)\mathcal{S}_R\mathcal{P}_H(u^R \cdot \nabla)u^R\|_{L^2} \lesssim \psi_N(k^R)\|b^R\|_{H^2}\|b^R\|_{L^\infty} \]
\[ \lesssim \psi_N(k^R)(\|b^R\|_{L^2}^\frac{\lambda}{\lambda+2}\|b^R\|_{H^\lambda}^\frac{\lambda}{\lambda+2}\|b^R\|_{L^\infty}^\frac{\lambda}{\lambda+2}) \lesssim c(N)(1 + \|b^R\|_{L^2}^2) \]
where we used Gagliardo-Nirenberg’s inequalities and \((20)\). Hence, along with \((11)\), a standard theory of stochastic differential equations (e.g. \([32, 42]\)), we see that the solution exists on \([0, T]\) for some \( T = T(N) > 0 \). As we will see in Proposition 4.3, we will be able to show that for any \( T > 0 \), the solution exists and lies in the space \( L^{p_0}(\Omega; \mathcal{L}_N([0, T]; H^2(\mathbb{R}^3))) \) where \( p_0 \in [4, \infty) \) is such that \( X_0 \in L^{p_0}(\Omega; H^2(\mathbb{R}^3)) \) by hypothesis of Theorem 2.1. Concerning path-wise uniqueness of such a solution, remarkably, the Hall term already creates an obstacle that does not exist for the NSE type nonlinear term. Indeed, a standard proof of path-wise uniqueness requires a computation such as this (see \((26)\) with \( s = 1 = 0)\):
\[ \|(\mathcal{S}_R[(u^R_1 \cdot \nabla)u^R_1] - \mathcal{S}_R[(u^R_2 \cdot \nabla)u^R_2], u^R_1 - u^R_2)\| \]
\[ = \|((u^R_1 - u^R_2) \cdot \nabla)u^R_1 + (u^R_2 - u^R_1) \cdot \nabla)(u^R_1 - u^R_2), \mathcal{S}_R(u^R_1 - u^R_2)\| \lesssim \|u^R_1 - u^R_2\|_{H^s} \lesssim \|u^R_1 - u^R_2\|_{L^2} \]
due to \((1c)\), Hölder’s inequality and the hypothesis that \( s > \frac{7}{2} \). However, an analogous attempt at Hall term proves to be not so straight-forward, unless we are willing to give up a constant that depends on \( R \):
\[ \|((j^R_1 \times b^R_1) - j^R_2 \times b^R_2) - \mathcal{S}_R\Phi(\nabla \times (j^R_1 \times b^R_1), j^R_2 \times b^R_2)\| \]
\[ = \int [(b^R_1 - b^R_2) \cdot \nabla b^R_1 + (b^R_2 \cdot \nabla)(b^R_1 - b^R_2)] \cdot \nabla \times (b^R_1 - b^R_2) \]
\[ \lesssim (\|b^R_1 - b^R_2\|_{L^\infty}\|\nabla b^R_1\|_{L^\infty} + \|b^R_2\|_{L^\infty}\|\nabla(b^R - b^R_2)\|_{L^\infty})\|\nabla \times (b^R_1 - b^R_2)\|_{L^2} \]
\[ \lesssim (R + \|b^R_1\|_{H^s} + \|b^R_2\|_{H^s})\|b^R - b^R_2\|_{L^2} \]
where we used integration by parts, \((16b)\), Hölder’s inequality and the truncation. Subsequently we will also need to prove the path-wise uniqueness of the solution to the Hall-MHD system with the cutoff \((21)\) and hence such dependence on the truncation is not ideal. Intrinsically this difficulty is related to the fact that handling this Hall term requires help from magnetic diffusion. Nevertheless, considering the regularity of the solution that we delay to show until Proposition 4.3, we may show the path-wise uniqueness directly. Because its computations display clearly
Suppose that Proposition 4.1. cutoff function (see (27), (28)). Then it will be necessary to estimate the typical proof of path-wise uniqueness of such solutions (e.g. [52, Theorem 2.2]), (18b), (18c) in the following computations. We also comment that in contrast to the identical proof of the following Proposition 4.1 goes through 5 of the proof of [52, Theorem 2.2]). We remark ahead for the subsequent proof in elaborate in this endeavor and state it formally as Proposition 4.1 (this follows Step the unique techniques that are needed to estimate the Hall term, let us already subsequently to deduce

**Proposition 4.1.** Suppose that $X^R_1 \triangleq (u^R_1, b^R_1)$, $X^R_2 \triangleq (u^R_2, b^R_2) \in L^{p_0}(\Omega; L^{\infty}(0, T; H^s(\mathbb{R}^3)))$ for $s \geq \frac{7}{5}, p_0 \in [4, \infty)$ are two solutions to the truncated Hall-MHD system with cutoff [22a)-(22b)]. Then $X^R_1(t) \equiv X^R_2(t)$ for all $t \in [0, T] \ P$-almost surely.

**Proof of Proposition 4.1.** Let us suppose that $X^R_1 = (u^R_1, b^R_1)$, $X^R_2 = (u^R_2, b^R_2) \in L^{p_0}(\Omega; L^{\infty}(0, T; H^s(\mathbb{R}^3)))$ for $s \geq \frac{7}{5}, p_0 \in [4, \infty)$ are two solutions to the truncated Hall-MHD system with cutoff [22a)-(22b)]. Then we define

$$\tau_M \triangleq \inf\{t \geq 0 : \|X^R_1(t)\|_{H^s} \geq M\} \wedge \inf\{t \geq 0 : \|X^R_2(t)\|_{H^s} \geq M\}. \quad (23)$$

Thus, $T \wedge \tau_M \to T \ P$-almost surely as $M \to \infty$ because $X^R_1, X^R_2 \in L^{\infty}(0, T; H^s(\mathbb{R}^3)) \ P$-almost surely.

Now for $i \in \{1, 2\}$, $k^R_i = \|X^R_i\|_{H^s}, i \in (\frac{5}{2}, s - 1)$, we denote $\pi^R_i \triangleq u^R_i - u^R_2, b^R_i \triangleq b^R_i - b^R_2, \bar{X}^R_i \triangleq (\pi^R_i, b^R_i), j^R_i \triangleq \nabla \times b^R_i, \tilde{j}^R_i \triangleq j^R_i - j^R_2$ and use (22a) to obtain

$$dX^R_i = [-A\bar{X}^R_i + \mathcal{S}_R(\psi_N(k^R_i)[-B(X^R_1) - HX^R_2] - \psi_N(k^R_i)[-B(X^R_2) - HX^R_2])]dt$$
$$+ \mathcal{S}_R[\Phi(X^R_1(t)) - \Phi(X^R_2(t))]dW$$
$$+ \int_{\mathbb{Z}} [\mathcal{S}_R[\gamma(X^R_1(t-), z) - \gamma(X^R_2(t-), z)]\hat{N}(dt, dz)]. \quad (24)$$

We apply $J^{s-1}$ and Ito’s formula (e.g. [39] Theorem 5.1), [1] Theorem 4.4.7)) with $f(t, x) = x^2$; integrate the resulting equation over $\mathbb{R}^3$, and apply Ito’s product formula ([1] Theorem 4.4.13)) for a constant $c_0$ sufficiently large to be determined subsequently to deduce

$$e^{-c_0 \sum_{k=1}^4 \sum_{j=1}^4 (\mathcal{J}^R)^{\wedge} \cdot \tau_M \cdot \|X^R_1\|_{H^s} \cdot \mathcal{J}^R \cdot \|X^R_2(t \wedge \tau_M)\|_{H^s}^2} \leq (I + II + III + IV)(t) \quad (25)$$
where

\[ I \triangleq \int_0^{\tau^*} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 \int_0^\tau \| X_i^R \|_{H^s}^2 \, d\xi} \]

\[ \times \left[ -c_0 \sum_{k=1}^4 \sum_{i=1}^2 \| X_i^R \|_{H^s} \| X_i^R \|_{H^{s-1}}^2 \right. \]

\[ + 2(J^{s-1}X^R, [A J^{s-1} X^R] + J^{s-1}(\psi_N(k^R)] - B(X^R_i - H X^R_i) \]

\[ - \psi_N(k^R_i) [-B(X^R_i) - H X^R_i]) \]

\[ + \sum_{j=1}^\infty \| S_R[\Phi_1(X^R_1) - \Phi_1(X^R_2)]e_j \|_{H^{s-1}}^2 + \| S_R[\Phi_1(X^R_1) - \Phi_1(X^R_2)]e_j \|_{H^{s-1}}^2 \right] d\delta, \]

\[ II \triangleq \int_0^{\tau^*} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 \int_0^\tau \| X_i^R \|_{H^s}^2 \, d\xi} \]

\[ \times \sum_{j=1}^\infty 2(J^{s-1}X^R, J^{s-1}S_R[\Phi_1(X^R_1) - \Phi_1(X^R_2)]e_j) d\beta_j^2 \]

\[ + \int_0^{\tau^*} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 \int_0^\tau \| X_i^R \|_{H^s}^2 \, d\xi} \]

\[ \times \sum_{j=1}^\infty 2(J^{s-1}X^R, J^{s-1}S_R[\Phi_2(X^R_1) - \Phi_2(X^R_2)]e_j) d\beta_j^2, \]

\[ III \triangleq \int_0^{\tau^*} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 \int_0^\tau \| X_i^R \|_{H^s}^2 \, d\xi} \]

\[ \times \int_Z \| S_R[\gamma(X^R(\delta-), z) - \gamma(X^R(\delta-), z)] \|_{H^{s-1}}^2 \mathcal{N}(d\delta, dz), \]

\[ IV \triangleq \int_0^{\tau^*} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 \int_0^\tau \| X_i^R \|_{H^s}^2 \, d\xi} \]

\[ \times \int_Z 2(J^{s-1}X^R, J^{s-1}S_R[\gamma(X^R(\delta-), z) - \gamma(X^R(\delta-), z)] \mathcal{N}(d\delta, dz). \]
We first work on $I$: we rewrite

\[
2(J^{s-1}\mathcal{X}^R, [-AJ^{s-1}\mathcal{X}^R + J^{s-1}(\psi_N(k_1^R)[-B(X_1^R) + H X_1^R] - \psi_N(k_2^R)[-B(X_2^R) + H X_2^R]])
\]

\[
= -2\eta\|\nabla b^R\|^2_{H^{s-1}} - 2(\psi_N(k_1^R) - \psi_N(k_2^R)) \int J^{s-1}(u_1^R \cdot \nabla u_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
- 2\psi_N(k_2^R) \int J^{s-1}(\nabla u_1^R \cdot J^{s-1}\mathcal{X}^R)
\]

\[
- 2\psi_N(k_2^R) \int J^{s-1}(\nabla u_2^R \cdot J^{s-1}\mathcal{X}^R)
\]

\[
+ 2(\psi_N(k_1^R) - \psi_N(k_2^R)) \int J^{s-1}(b_1^R \cdot \nabla b_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
+ 2\psi_N(k_2^R) \int J^{s-1}(b_1^R \cdot \nabla b_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
- 2(\psi_N(k_1^R) - \psi_N(k_2^R)) \int J^{s-1}(u_2^R \cdot \nabla b_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
- 2\psi_N(k_2^R) \int J^{s-1}(u_2^R \cdot \nabla b_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
+ 2(\psi_N(k_1^R) - \psi_N(k_2^R)) \int J^{s-1}(b_1^R \cdot \nabla u_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
+ 2\psi_N(k_2^R) \int J^{s-1}(b_1^R \cdot \nabla u_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
- 2(\psi_N(k_1^R) - \psi_N(k_2^R)) \int J^{s-1}(\nabla \times (j_1^R \times b_1^R)) \cdot J^{s-1}\mathcal{X}^R
\]

\[
- 2\psi_N(k_2^R) \int J^{s-1}\nabla \times (j_2^R \times b_1^R) \cdot J^{s-1}\mathcal{X}^R
\]

\[
- 2\psi_N(k_2^R) \int J^{s-1}\nabla \times (j_2^R \times b_1^R) \cdot J^{s-1}\mathcal{X}^R \triangleq \sum_{i=1}^{16} I_i.
\]

Firstly, we estimate $\sum_{i\in\{2,5,8,11,14\}} I_i = I_2 + I_5 + I_8 + I_{11} + I_{14}$. To do so, we observe that due to (20),

\[
|\psi_N(k_1^R) - \psi_N(k_2^R)| \lesssim \|\mathcal{X}^R\|_{H^{s-1}}.
\]
because \( k^i = \| X^R_i \|_{H^s} \) for \( i \in (\frac{5}{2}, s - 1) \). Now we estimate from \([26]\)

\[
\sum_{i \in \{2, 5, 8, 11, 14\}} I_i \leq \| X^R_{H^{s-1}}(u^1_H \cdot \nabla u^1_H) \|_{H^{s-1}} + \| b^R_1 \cdot \nabla b^R_1 \|_{H^{s-1}} + \| b^R_1 \cdot \nabla u^1_H \|_{H^{s-1}} \| \tau^R \|_{H^{s-1}} + \\
+ \| j^R_1 \times b^R_1 \|_{H^{s-1}} \| J^{s-1} \nabla b^R \|_{L^2})
\]

(28)

where for one of the Hall terms, we used integration by parts so that

\[
\int J^{s-1} (\nabla \times (j^R_1 \times b^R_1)) \cdot J^{s-1} b^R = \int J^{s-1} (j^R_1 \times b^R_1) \cdot J^{s-1} \nabla \times b^R,
\]

the hypothesis that \( s > \frac{7}{2} \) and Young’s inequality. Next, we estimate from \([26]\)

\[
\sum_{i \in \{3, 6, 9, 12\}} I_i \leq \| (\tau^R \cdot \nabla) u^1_H \|_{H^{s-1}} \| \tau^R \|_{H^{s-1}} + \| (b^R \cdot \nabla) b^R_1 \|_{H^{s-1}} \| \tau^R \|_{H^{s-1}} + \\
+ \| (\tau^R \cdot \nabla) b^R_1 \|_{H^{s-1}} \| b^R \|_{H^{s-1}} + \| (b^R \cdot \nabla) u^1_H \|_{H^{s-1}} \| b^R \|_{H^{s-1}} \| - 3)\]

(29)

by Hölder’s inequality, and the hypothesis that \( s > \frac{7}{2} \). Next, we estimate one of the Hall terms from \([26]\) as follows:

\[
I_{15} = -2 \psi_N(k^R_2) \int [J^{s-1}(J^R \times b^R_1) - (J^{s-1} j^R_1) \times b^R_1] \cdot J^{s-1} j^R_1 \leq \\
\leq \| J^{s-1}(J^R \times b^R_1) - (J^{s-1} j^R_1) \times b^R_1 \|_{L^2} \| J^{s-1} j^R_1 \|_{L^2} \leq \\
\leq \| J^R \|_{H^{s-1}} \| \nabla b^R_1 \|_{L^\infty} + \| J^R \|_{L^\infty} \| b^R \|_{H^{s-1}} \| \nabla J^{s-1} b^R \|_{L^2} \leq \\
\leq \| J^R \|_{H^{s-1}} \| X^R \|_{H^s} + \| b^R \|_{H^{s-1}} \| X^R \|_{H^s} \| \nabla J^{s-1} b^R \|_{L^2} \leq \\
\leq \frac{2}{3} \| \nabla b^R \|_{H^{s-1}} + c \| X^R \|_{H^s} \| \nabla b^R \|_{H^{s-1}}
\]

(30)
due to integration by parts, (16c) Lemma 3.1, the hypothesis that \( s > \frac{7}{2} \) and Young’s inequality. Next, we estimate from (26)

\[
\sum_{i\in\{4,7,10,13\}} \lesssim (\|\nabla u_2\|_{L^\infty} \|J^{s-2}\nabla R\|_{L^2} + \|J^{s-1}u_2\|_{L^2}\|\nabla R\|_{L^\infty})\|\nabla R\|_{H^{s-1}} \\
+ (\|\nabla u_2^R\|_{L^\infty} \|J^{s-2}\nabla R\|_{L^2} + \|J^{s-1}u_2^R\|_{L^2}\|\nabla R\|_{L^\infty})\|\nabla R\|_{H^{s-1}} \\
+ (\|\nabla u_2^R\|_{L^\infty} \|J^{s-2}\nabla R\|_{L^2} + \|J^{s-1}u_2^R\|_{L^2}\|\nabla R\|_{L^\infty})\|\nabla R\|_{H^{s-1}} \leq \|X^R\|_{H^s} \|\nabla R\|_{H^{s-1}}
\]

where we used (1c), Hölder’s inequality, Lemma 3.1 and the hypothesis that \( s > \frac{7}{2} \).

Finally, we estimate one of the Hall terms from (26) as follows:

\[
I_{16} = -2\psi_N(k^R_2) \int J^{s-1}(j^R_2 \times \overline{b}^R) \cdot J^{s-1}\nabla \times \overline{b}^R \\
\leq 2\|j^R_2 \times \overline{b}^R\|_{H^{s-1}} \|J^{s-1}\nabla \overline{b}^R\|_{L^2} \\
\lesssim \|j^R_2\|_{H^s} \|\overline{b}^R\|_{H^{s-1}} \|J^{s-1}\nabla \overline{b}^R\|_{L^2} \leq \frac{2}{3} \|\nabla \overline{b}^R\|_{H^{s-1}}^2 + c\|X^R\|_{H^s}^2 \|\overline{X}^R\|_{H^{s-1}}^2
\]

due to integration by parts, Hölder’s inequality, the hypothesis that \( s > \frac{7}{2} \) and Young’s inequality. Within \( I \) from (25), we may also estimate

\[
\sum_{j=1}^{\sum} \|S_R[\Phi_1(X^R_1) - \Phi_1(X^R_2)]e_j\|_{H^{s-1}}^2 + \|S_R[\Phi_2(X^R_1) - \Phi_2(X^R_2)]e_j\|_{H^{s-1}}^2 \leq \|\overline{X}^R\|_{H^{s-1}}^2 (33)
\]

by (18a) and (12). Thus, we conclude that due to (28) - (33) applied to (26),

\[
I \leq \int_0^{t+\tau_M} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_i^\delta \|X^R\|_{H^{s-1}}^4} \int \|\nabla \overline{b}^R\|_{H^{s-1}}^2 + c\|\overline{X}^R\|_{H^{s-1}}^2 d\delta
\]

if \( c_0 \) is sufficiently large. We now rely on Lemma 3.2. Let us denote \( Z(t) = 0 \),

\[
E(t) = \sup_{\delta \in [0,t]} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_i^\delta \|X^R\|_{H^{s-1}}^4} \int \|\nabla \overline{b}^R\|_{H^{s-1}}^2 d\delta, \\
Y(t) = \int_0^{t+\tau_M} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_i^\delta \|X^R\|_{H^{s-1}}^4} \int \|\nabla \overline{b}^R\|_{H^{s-1}}^2 d\delta, \\
so that applying (34) to (25), and then taking supremum over time on the right and then left sides give

\[
E(t) + \eta Y(t) \leq \int_0^t cE(\delta) d\delta + \sup_{\delta \in [0,t]} |II(\delta) + III(\delta) + IV(\delta)| \\
\leq \int_0^t \phi(\delta) E(\delta) d\delta + V(t)
\]

(36)
where $E(t), Y(t), V(t), \phi(t)$ are all non-negative processes while $V(t)$ is clearly non-decreasing. By applying expectation $E$ on $V$, we obtain

$$E[V(t)] \leq E[ \sup_{\lambda \in [0, t]} \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi]$$

$$\quad \times \sum_{j=1}^{\infty} 2(J_s^{-1} \pi^R, J_s^{-1} S_H[\Phi_1(X_1^R) - \Phi_1(X_2^R)] e_j)$$

$$+ \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi$$

$$\quad \times \sum_{j=1}^{\infty} 2(J_s^{-1} b^R, J_s^{-1} S_H[\Phi_2(X_1^R) - \Phi_2(X_2^R)] e_j)$$

$$+ E[ \sup_{\lambda \in [0, t]} \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi]$$

$$\quad \times \int \|S_R[\gamma(X_1^R, z) - \gamma(X_2^R, z)]\|_{H^s} d\xi$$

$$+ E[ \sup_{\lambda \in [0, t]} \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi]$$

$$\quad \times \int 2(J_s^{-1} X^R, J_s^{-1} S_R[\gamma(X_1^R, z) - \gamma(X_2^R, z)] N(\delta, d\xi)) \Delta (V_1 + V_2 + V_3)$$

due to \eqref{36} and \eqref{25}. We now estimate the three terms as follows. For the first term, we may compute

$$V_1 \leq E[ \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi]$$

$$\quad \times \sum_{j=1}^{\infty} [(J_s^{-1} \pi^R, J_s^{-1} S_H[\Phi_1(X_1^R) - \Phi_1(X_2^R)] e_j)$$

$$+ (J_s^{-1} b^R, J_s^{-1} S_H[\Phi_2(X_1^R) - \Phi_2(X_2^R)] e_j)]^2 d\xi]$$

$$\leq E[ \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi]$$

$$\quad \times (\|\pi^R\|_{H^{-1}}^2 \|X_R^R\|_{H^s}^2 + \|b^R\|_{H^{-1}}^2 \|X_R^R\|_{H^s}^2 d\xi)^\frac{1}{2}$$

$$\leq E[ \sup_{\delta \in [0, t]} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi]$$

$$\quad \times \left( \int_0^{\lambda \wedge \tau_M} e^{-c_0 \sum_{k=1}^{4} \sum_{i=1}^{2} f_{0}^k \|X_R^k\|_{H^s}} d\xi \|X_R^R\|_{H^s} \delta M(\delta, \tau_M) \right)^\frac{1}{2}$$

$$\leq \beta E [E(t)] + c \int_0^t E [E(\delta)] d\delta$$

for any $\beta > 0$, where we used \eqref{37}. Burkholder-Davis-Gundy inequality (e.g. \cite{24} Proposition 2.4 (i)), \cite{25} pg. 160), Hölder’s inequality, \cite{12}, Young’s inequality and
For the second term, we estimate

\[ V_2 \leq \mathbb{E}\left[ \int_0^{t \wedge \tau_M} \int_Z e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_k^i \|X_i^R\|_{H^{s-1}}} \, d\xi \right. \]

\[ \times \|S_R[\gamma(X_1^R(\delta), z) - \gamma(X_2^R(\delta), z)]\|_{H^{s-1}}^2 \lambda(dz) \, d\delta \]

\[ \leq \mathbb{E}\left[ \int_0^{t \wedge \tau_M} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_k^i \|X_i^R\|_{H^{s-1}}} \|X_i^R(\delta)\|_{H^{s-1}}^2 \, d\delta \right] \leq \int_0^t \mathbb{E}[E(\delta)] \, d\delta \]

by (37), (18a), (12) and (35a). For the third term,

\[ V_3 \leq \mathbb{E}\left[ \left( \int_0^{t \wedge \tau_M} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_k^i \|X_i^R\|_{H^{s-1}}} \|X_i^R(\delta \wedge \tau_M)\|_{H^{s-1}}^2 \right) \frac{1}{2} \right] \]

\[ \leq \mathbb{E}\left[ \left( \sup_{\delta \in [0,t]} \left( \int_0^{t \wedge \tau_M} e^{-c_0 \sum_{k=1}^4 \sum_{i=1}^2 f_k^i \|X_i^R\|_{H^{s-1}}} \|X_i^R(\delta \wedge \tau_M)\|_{H^{s-1}}^2 \right) \frac{1}{2} \right] \]

\[ \leq \beta \mathbb{E}[E(t)] + c \int_0^t \mathbb{E}[E(\delta)] \, d\delta \]

for any \( \beta > 0 \) by (37), Burkholder-Davis-Gundy inequality, (18a), (12), Young’s inequality and (35a). Therefore,

\[ \mathbb{E}[V(t)] \leq 2\beta \mathbb{E}[E(t)] + c \int_0^t \mathbb{E}[E(\delta)] \, d\delta \] (41)

for any \( \beta > 0 \) due to (38) - (40) applied to (37). After applying Lemma 3.2, and taking \( M \to \infty \) so that \( T \wedge \tau_M \to T \), we obtain \( \|X_i^R\|_{H^{s-1}} \equiv 0 \) \( \mathbb{P} \)-almost surely. \( \square \)

Next, before we prove the \( H^s(\mathbb{R}^3) \)-bound of the solution to the truncated Hall-MHD system with the cutoff (22a) - (22b), we first prove the \( L^2(\mathbb{R}^3) \)-energy bound; we remark ahead that the identical proof goes through for that without the truncation.

**Proposition 4.2.** Under the hypothesis of Theorem 2.1, the solution to the truncated Hall-MHD system with the cutoff (22a) - (22b) over \([0, T]\) satisfies the following bounds:

\[ \mathbb{E}[\|X_i^R(t)\|_{L^2}^2] + 2\eta \mathbb{E}\left[ \int_0^t \|\nabla b R\|_{L^2}^2 \, d\xi \right] \leq (1 + \mathbb{E}[\|X_0\|_{L^2}^2]) e^{cT} \quad \forall \ t \in [0, T], \] (42a)

\[ \mathbb{E}\left[ \sup_{t \in [0, T]} \|X_i^R(t)\|_{L^2}^2 \right] + 4\eta \mathbb{E}\left[ \int_0^T \|\nabla b R\|_{L^2}^2 \, d\xi \right] \leq (2\mathbb{E}[\|X_0\|_{L^2}^2] + 1) e^{cT}. \] (42b)

**Proof of Proposition 4.2.** We consider the truncated Hall-MHD system with the cutoff (22a) - (22b), apply Itô’s formula with \( f(t, x) = x^2 \), integrate over \( \mathbb{R}^3 \) to
obtain for $\tau^R_M \triangleq \inf\{t \geq 0 : \|X^R(t)\|_{L^2} \geq M\}$,
\[
\|X^R(t \wedge \tau^R_M)\|_{L^2}^2 = \|X^R(0)\|_{L^2}^2 + \int_0^{t \wedge \tau^R_M} 2 \left( X^R(\delta-) \left[ -AX^R + \psi_N(k^R)S_R[-B(X^R) - HX^R]\right] \right) d\delta \\
+ \int_0^{t \wedge \tau^R_M} 2 \left( X^R(\delta-) , S_R \Phi(X^R(\delta)) \right) dW(\delta) \\
+ \sum_{j=1}^{\infty} \int_0^{t \wedge \tau^R_M} \|S_R \Phi_1(X^R)e_j\|_{L^2}^2 + \|S_R \Phi_2(X^R)e_j\|_{L^2}^2 d\delta \\
+ \int_0^{t \wedge \tau^R_M} \int_Z 2(X^R(\delta-) , S_R \gamma(X^R(\delta-), z)) + \|S_R \gamma(X^R(\delta-), z)\|_{L^2}^2 \tilde{N}(d\delta, dz) \\
+ \int_0^{t \wedge \tau^R_M} \|S_R \gamma(X^R(\delta-), z)\|_{L^2}^2 \lambda(dz) d\delta
\]
\[
\triangleq \|X^R(0)\|_{L^2}^2 + VI + VII + VIII + IX + X.
\]

We compute for $VI$,
\[
(X^R(\delta-) \left[ -AX^R + \psi_N(k^R)S_R[-B(X^R) - HX^R]\right] \\
= -\eta \|\nabla b^R\|_{L^2}^2 - \psi_N(k^R) \int (u^R \cdot \nabla) \frac{|u^R|^2}{2} + (u^R \cdot \nabla) \frac{|b^R|^2}{2} \\
+ \psi_N(k^R) \int (b^R \cdot \nabla)(u^R \cdot b^R) - \psi_N(k^R) \int \|[(\nabla \times b^R) \times b^R] \cdot \nabla \times b^R\|_{L^2}^2 \\
= -\eta \|\nabla b^R\|_{L^2}^2
\]
by integration by parts, (16c) and (1c). Next, we estimate for $VIII$ and $X$,
\[
\sum_{j=1}^{\infty} \|S_R \Phi_1(X^R)e_j\|_{L^2}^2 + \|S_R \Phi_2(X^R)e_j\|_{L^2}^2 \\
+ \int_Z \|\gamma(X^R(\delta-), z)\|_{L^2}^2 \lambda(dz)
\]
\[
\leq \sum_{i=1}^{2} \|\Phi_i(X^R)\|_{L^2}^2 \lambda(d\delta) + \int_Z \|\gamma(X^R(s-), z)\|_{L^2}^2 \lambda(dz)
\]
by (18a), (7) and (11). Finally, both $VII$ and $IX$ are local martingales with zero expectation (e.g. [132 pg. 196 and Proposition 3.12]). Therefore, after applying (44) and (45) in (43), taking expectation $\mathbb{E}$, using that $\mathbb{E}[\|X^R(0)\|_{L^2}^2] \leq \mathbb{E}[\|X_0\|_{L^2}^2]$ due to (18a), Gronwall’s inequality type argument leads to
\[
\mathbb{E}[\|X^R(t \wedge \tau^R_M)\|_{L^2}^2] + 2\eta \mathbb{E}\left[\int_0^{t \wedge \tau^R_M} \|\nabla b^R\|_{L^2}^2 d\delta\right] \leq (1 + \mathbb{E}[\|X_0\|_{L^2}^2]) e^{ct}.
\]
Now we may write
\[
\mathbb{E}[\|X^R(t \wedge \tau^R_M)\|_{L^2}^2] = \mathbb{E}[\|X^R(t)\|_{L^2}^2 \chi_{(\tau^R_M < t)}] + \mathbb{E}[\|X^R(t)\|_{L^2}^2 \chi_{(\tau^R_M \geq t)}].
\]
From the right continuity of $X^R$, we know $\|X^R(\tau^R_M)\|_{L^2} \geq M \mathbb{P}$-a.s. so that
\[
\mathbb{E}[\|X^R(t \wedge \tau^R_M)\|_{L^2}^2] \geq \mathbb{E}[\|X^R(\tau^R_M)\|_{L^2}^2 \chi_{(\tau^R_M < t)}] \geq M^2 \mathbb{P}(\{\omega \in \Omega : \tau^R_M < t\})
\]
due to (47). Therefore,
\[
\lim_{M \to \infty} \mathbb{P}(\{\omega \in \Omega : \tau^R_M < t\}) \leq \lim_{M \to \infty} \frac{1}{M^2} (1 + \mathbb{E}[\|X_0\|_{L^2}^2]) e^{ct} = 0
\]
by Chebyshev’s inequality and (46). Hence, \( t \wedge \tau^R_M \to t \) as \( M \to \infty \) \( P \)-almost surely so that taking \( M \to \infty \) in (46) deduces (42a). Next, we return to (43), use the previous computations of (44), (45), take supremum over \( t \in [0, T] \) on the right and then left sides, and then expectation \( \mathbb{E} \) to deduce

\[
\mathbb{E}[\sup_{t \in [0, T \wedge \tau^R_M]} \|X^R(t)\|^2_{L^2_z} + 2\eta \mathbb{E}[\int_0^{T \wedge \tau^R_M} \|\nabla b^R\|^2_{L^2_t} dt] \\
\leq \mathbb{E}[\|X^R(0)\|^2_{L^2_z}] + c\mathbb{E}\left[\int_0^{T \wedge \tau^R_M} \left\|S_{R}\gamma(X^R(\delta-), z)\right\|^2_{L^2_z} N(\delta, dz)\right] \\
+ \mathbb{E}\left[\sup_{t \in [0, T \wedge \tau^R_M]} \left|\int_0^t 2(X^R(\delta-), S_{R}\Phi(X^R(\delta)) dW(\delta))\right|\right] \\
+ \mathbb{E}\left[\sup_{t \in [0, T \wedge \tau^R_M]} \left|\int_0^t 2(X^R(\delta-), S_{R}\gamma(X^R(\delta-), z))\right| N(\delta, dz)]\right]
\]

(50)

We estimate from (50)

\[
XI \lesssim \mathbb{E}\left[\int_0^{T \wedge \tau^R_M} \left\|\gamma(X^R(\delta-), z)\right\|^2_{L^2_2} \lambda(dz) d\delta\right] \lesssim \mathbb{E}\left[\int_0^{T \wedge \tau^R_M} 1 + \|X^R\|^2_{L^2_z} d\delta\right]
\]

(51)

by (18a) and (11). Next, we estimate from (50)

\[
XII \lesssim \sum_{j=1}^{\infty} \mathbb{E}\left[\left(\int_0^{T \wedge \tau^R_M} \|u^R\|^2_{L^2_z} \left\|S_{R}\Phi_1(X^R)e_j\right\|^2_{L^2_z} d\delta\right)^{\frac{1}{2}}\right]
\\
+ \mathbb{E}\left[\left(\int_0^{T \wedge \tau^R_M} \|b^R\|^2_{L^2_z} \left\|S_{R}\Phi_2(X^R)e_j\right\|^2_{L^2_z} d\delta\right)^{\frac{1}{2}}\right]
\]

(52)

\[
\lesssim \frac{1}{4} \mathbb{E}\left[\sup_{t \in [0, T \wedge \tau^R_M]} \|X^R(t)\|^2_{L^2_z} + c\mathbb{E}\left[\int_0^{T \wedge \tau^R_M} 1 + \|X^R\|^2_{L^2_z} dt\right]\right]
\]

by Burkholder-Davis-Gundy inequality, (18a), (11), and Young’s inequality. Similarly, we may estimate from (50)

\[
XIII \lesssim \mathbb{E}\left[\left(\int_0^{T \wedge \tau^R_M} \left\|X^R(\delta-), z\right\|^2_{L^2_z} \left\|\gamma(X^R(\delta-), z)\right\|^2_{L^2_z} d\delta\right)^{\frac{1}{2}}\right]
\]

(53)

\[
\lesssim \frac{1}{4} \mathbb{E}\left[\sup_{t \in [0, T \wedge \tau^R_M]} \|X^R(t)\|^2_{L^2_z} + c\mathbb{E}\left[\int_0^{T \wedge \tau^R_M} 1 + \|X^R(\delta-)\|^2_{L^2_z} d\delta\right]\right]
\]

by Burkholder-Davis-Gundy inequality, (18a), (11), and Young’s inequality. After applying (51) - (53) to (50), and subtracting \( \frac{1}{2} \mathbb{E}[\sup_{t \in [0, T \wedge \tau^R_M]} \|X^R(t)\|^2_{L^2_z}] \) from both
sides, Gronwall’s inequality leads to

\[
\mathbb{E}[\sup_{t \in [0,T \wedge \tau_M^R]} \|X^R(t)\|_{H^s}^2] + 4\eta \mathbb{E}\left[ \int_0^{T \wedge \tau_M^R} \|\nabla b^R\|_{L^2}^2 \, d\delta \right] \leq (2\mathbb{E}[\|X_0\|_{H^s}^2] + 1)e^{cT}. \tag{54}
\]

Similarly to (47) - (49), we may let \( M \to \infty \) in (54) so that \( T \wedge \tau_M^R \to T \) \( \mathbb{P} \)-a.s., and therefore we obtain (42b). This completes the proof of Proposition 4.2. \( \square \)

**Proposition 4.3.** Under the hypothesis of Theorem 2.1, the solution to the truncated Hall-MHD system with the cutoff \( (22a) - (22b) \) over \([0,T]\) satisfies the following bounds: for \( p \in [1, \frac{4}{p_0}] \) where \( p_0 \in (4, \infty) \) is such that \( X_0 \in L^{p_0}(\Omega; H^s(\mathbb{R}^3) \cap H^r) \),

\[
\mathbb{E}[\|X^R(t)\|_{H^s}^{2p}] + \eta p \mathbb{E}\left[ \int_0^t \|\nabla b^R\|_{H^r}^2 \|X^R\|_{H^s}^{2p-2} \, d\delta \right] \tag{55a}
\]

\[
\leq (1 + \mathbb{E}[\|X_0\|_{H^s}^{2p}])e^{p(1+c_N)t}, \quad \forall \, t \in [0,T],
\]

\[
\mathbb{E}[\sup_{t \in [0,T]} \|X^R(t)\|_{H^s}^{2p}] + 2\eta p \mathbb{E}\left[ \int_0^T \|\nabla b^R\|_{H^r}^2 \|X^R\|_{H^s}^{2p-2} \, d\delta \right] \tag{55b}
\]

\[
\leq (1 + 2\mathbb{E}[\|X_0\|_{H^s}^{2p}])e^{(c_p+1)c_N T},
\]

\[
\mathbb{E}\left[ \left( \int_0^T \|\nabla b^R\|_{H^r}^2 \, d\delta \right)^{\frac{p}{2}} \right] \lesssim_{p_0,T,N} 1. \tag{55c}
\]

**Proof of Proposition 4.3.** We set \( \tau_M^R \triangleq \inf\{t \geq 0 : \|X^R(t)\|_{H^s} > M\} \). We consider the truncated stochastic Hall-MHD system with the cutoff \( (22a) - (22b) \), apply \( J^s \), apply Itô’s formula with \( f(t,x) = x^2 \), integrate over \( \mathbb{R}^3 \), apply Itô’s formula again with \( f(t,x) = x^p \) to obtain

\[
\|X^R(t \wedge \tau_M^R)\|_{H^s}^{2p} - \|X^R(0)\|_{H^s}^{2p} \leq \int_0^{t \wedge \tau_M^R} p\|X^R\|_{H^s}^{2(p-1)}(2J^sX^R, -AJ^sX^R + \psi_N(kR)S_RJ^s[-B(X^R) -HX^R]) \, d\delta
\]

\[
+ \int_0^{t \wedge \tau_M^R} p(2p-1)\|X^R(\delta)\|_{H^s}^{2(p-1)}\|S_RJ^s\Phi(X^R(\delta))\|_{L^2}^2 \, d\delta
\]

\[
+ \int_0^{t \wedge \tau_M^R} 2p\|X^R(\delta)\|_{H^s}^{2(p-1)}(J^sX^R, S_RJ^s\Phi(X^R(\delta))dW)
\]

\[
+ \int_0^{t \wedge \tau_M^R} \int_2 \|J^sX^R + S_RJ^s\gamma(X^R(\delta-),z)\|_{L^2}^{2p} - \|X^R(\delta)\|_{H^s}^{2p}, \tilde{N}(d\delta, dz)
\]

\[
+ \int_0^{t \wedge \tau_M^R} \int_2 \|J^sX^R(\delta) + S_RJ^s\gamma(X^R(\delta-),z)\|_{L^2}^{2p} - \|X^R\|_{H^s}^{2p},
\]

\[
- 2p(J^sX^R, S_RJ^s\gamma(X^R(\delta-),z))\|X^R(\delta)\|_{H^s}^{2(p-1)}\lambda(dz) \, d\delta.
\]

\( \square \)
where the inequality is due to Hölder’s inequality. We can estimate

\[
(2J^s X^R, -AJ^s X^R + \psi_N(k^R)S R J^s [-B(X^R) - H X^R])
\]

\[
\leq -2\eta \|J^s \nabla b R\|^2_{2^p, H^s} + \eta \|\nabla b R\|^2_{2^p, H^s} X^R \|2^p-2 d\delta
\]

\[
\leq \|X^R(0)\|^2_{2^p, H^s} + \eta p \int_0^{t_{A \tau_M}} \|\nabla b R\|^2_{2^p, H^s} X^R \|2^p-2 d\delta
\]

\[
+ p(2p-1) \int_0^{t_{A \tau_M}} \|X^R(\delta)\|^{2(p-1)} \|S_R J^s \Phi(X^R(\delta))\|^2_{L^2} d\delta
\]

\[
+ 2p \int_0^{t_{A \tau_M}} \|X^R(\delta)\|^{2(p-1)} (J^s X^R, S_R J^s \Phi(X^R(\delta)) dW)
\]

\[
+ \int_0^{t_{A \tau_M}} \int_Z \|J^s X^R + S_R J^s \gamma(X^R(\delta, z))\|^{2p} - \|X^R(\delta)\|^{2p}_{2^p, H^s}
\]

\[
- 2p(J^s X^R, S_R J^s \gamma(X^R(\delta, z))) ||X^R||^{2(p-1)}_{H^s} \mathcal{N}(d\delta, dz)
\]

\[
+ 2p \int_0^{t_{A \tau_M}} \int_Z (J^s X^R, S_R J^s \gamma(X^R(\delta, z))) ||X^R||^{2(p-1)}_{H^s} \mathcal{N}(d\delta, dz).
\]

Now we take expectation \(E\) to deduce

\[
E[\|X^R(t \wedge \tau_M)\|^{2p}_{2^p, H^s}] + \eta p E[\int_0^{t_{A \tau_M}} \|\nabla b R\|^2_{2^p, H^s} X^R \|2^p-2 d\delta]
\]

\[
\leq E[\|X^R(0)\|^2_{2^p, H^s}]
\]

\[
+ p c_N E\left[\int_0^{t_{A \tau_M}} \|X^R\|^{2p}_{2^p, H^s} d\delta\right]
\]

\[
+ p(2p-1) E\left[\int_0^{t_{A \tau_M}} \|X^R(\delta)\|^{2(p-1)} \|S_R J^s \Phi(X^R(\delta))\|^2_{L^2} d\delta\right]
\]

\[
+ E\left[\int_0^{t_{A \tau_M}} \int_Z \|J^s X^R + S_R J^s \gamma(X^R(\delta, z))\|^{2p} - \|X^R(\delta)\|^{2p}_{2^p, H^s}
\]

\[
- 2p(J^s X^R, S_R J^s \gamma(X^R(\delta, z))) ||X^R||^{2(p-1)}_{H^s} \lambda(dz)d\delta\right].
\]
where we used that both integrals involving \(dW\) and \(\tilde{N}(dt, dz)\) are local martingales. We next estimate

\[
\mathbb{E}\left[\int_0^{t \wedge \tau^B_M} \|X^R(\delta)\|_{H^s}^{2(p-1)} \|S_R J^s \Phi(X^R(\delta))\|_{L^2}^2 d\delta\right]
\]

\[\lesssim_p \mathbb{E}\left[\int_0^{t \wedge \tau^B_M} 1 + \|X^R(\delta)\|_{H^s}^{2p} d\delta\right]
\]

(59)

by (18a) and (11). For the estimate on the last term, we use Taylor’s formula to deduce

\[
\|X^R + S_R \gamma(X^R(\delta-), z)\|_{H^s}^{2p} - \|X^R(\delta)\|_{H^s}^{2p} - 2p(J^s X^R, S_R J^s \gamma(X^R(\delta-), z))\|X^R\|_{H^s}^{2(p-1)} \|X^R\|_{H^s}^{2(p-1)}
\]

\[\lesssim c_p (\|X^R\|_{H^s}^{2p-2} \|S_R \gamma(X^R(\delta-), z)\|_{H^s}^2 + \|S_R \gamma(X^R(\delta-), z)\|_{H^s}^{2p})
\]

and thus

\[
\mathbb{E}\left[\int_0^{t \wedge \tau^B_M} \int_Z \|J^s X^R + S_R J^s \gamma(X^R(\delta-), z)\|_{L^2}^{2p} - \|X^R(\delta)\|_{H^s}^{2p} d\delta\right]
\]

\[\lesssim_p \mathbb{E}\left[\int_0^{t \wedge \tau^B_M} 1 + \|X^R\|_{H^s}^{2p} d\delta\right]
\]

(60)

where we used (60), (18a) and (11). Applying this and (59) to (58), Gronwall’s inequality deduces (55a) after taking \(M \to \infty\). Next, we return to (57), take absolute values, supremum over \(t \in [0, T]\) on the right and then left sides, as well as expectation \(\mathbb{E}\) to deduce

\[
\mathbb{E}\left[\sup_{t \in [0, T]} \|X^R(t \wedge \tau^B_M)\|_{H^s}^{2p}\right] + \eta p \mathbb{E}\left[\int_0^{T \wedge \tau^B_M} \|\nabla h^R\|_{H^s}^2 \|X^R\|_{H^s}^{2p-2} d\delta\right]
\]

\[\lesssim \mathbb{E}[\|X^R(0)\|_{H^s}^{2p}] + p c_N \mathbb{E}\left[\int_0^{T \wedge \tau^B_M} \|X^R\|_{H^s}^{2p} d\delta\right]
\]

\[- 2p(J^s X^R, S_R J^s \gamma(X^R(\delta-), z))\|X^R\|_{H^s}^{2(p-1)} N'(d\delta, dz)d\delta]
\]

(62)

\[\lesssim p(2p - 1) \mathbb{E}\left[\int_0^{T \wedge \tau^B_M} \|X^R(\delta)\|_{H^s}^{2(p-1)} \|S_R J^s \Phi(X^R(\delta))\|_{L^2}^2 d\delta\right]
\]

\[+ 2p \mathbb{E}\left[\sup_{t \in [0, T]} \int_0^{t \wedge \tau^B_M} \|X^R(\delta)\|_{H^s}^{2(p-1)} (J^s X^R, S_R J^s \Phi(X^R(\delta))dW(\delta))\right]
\]

\[+ \mathbb{E}\left[\sup_{t \in [0, T]} \int_0^{T \wedge \tau^B_M} \int_Z \|J^s X^R + S_R \gamma(X^R(\delta-), z)\|_{L^2}^{2p} - \|X^R(\delta)\|_{H^s}^{2p} d\delta\right]
\]

\[\leq \mathbb{E}[\|X^R(0)\|_{H^s}^{2p}] + p c_N \mathbb{E}\left[\int_0^{T \wedge \tau^B_M} \|X^R\|_{H^s}^{2p} d\delta\right] + X V I + X V + X VI + X V I I .
\]
We first estimate

\[ X_{IV} \lesssim_p \mathbb{E}\left[ \int_0^{T \wedge \tau^R_M} 1 + \|X^R(\delta)\|_{H^2}^{2p} d\delta \right] \tag{63} \]

by (62), (18a) and (11). Next, we estimate

\[ X_{V} \lesssim_p \mathbb{E}\left( \int_0^{T \wedge \tau^R_M} \|X^R(\delta)\|_{H^2}^{4(p-1)} \|X^R\|_{H^2}^2 |\mathcal{S}_R J^* \Phi(X^R(\delta))|_{L^2}^2 d\delta \right)^{\frac{1}{2}} \tag{64} \]

\[ \leq \frac{1}{4} \mathbb{E}\left[ \sup_{t \in [0,T]} \|X^R(t \wedge \tau^R_M)\|_{H^2}^{2p} \right] + c \mathbb{E}\left[ \int_0^{T \wedge \tau^R_M} 1 + \|X^R(\delta)\|_{H^2}^{2p} d\delta \right] \]

by (62), Burkholder-Davis-Gundy inequality, (18a), (11), and Young’s inequality. Next, we estimate

\[ X_{VI} \lesssim_p \mathbb{E}\left[ \int_0^{T \wedge \tau^R_M} \int_Z \|X^R\|_{H^2}^{2p-2} |\mathcal{S}_R \gamma(X^R(\delta), z)|^2 d\delta \right] \tag{65} \]

\[ + \|\mathcal{S}_R \gamma(X^R(\delta), z)\|_{H^2}^{2p} \lambda(dz) d\delta \lesssim_p \mathbb{E}\left[ \int_0^{T \wedge \tau^R_M} 1 + \|X^R(\delta)\|_{H^2}^{2p} d\delta \right] \]

where we used (62), (60), (18a) and (11). Finally, we estimate

\[ X_{VII} \lesssim_p \mathbb{E}\left( \int_0^{T \wedge \tau^R_M} \int_Z \|J^* X^R\|_{L^2}^2 \mathcal{S}_R J^* \gamma(X^R(\delta), z) \|X^R\|_{H^2}^{4(p-1)} \lambda(dz) d\delta \right)^{\frac{1}{2}} \tag{66} \]

\[ \leq \frac{1}{4} \mathbb{E}\left[ \sup_{t \in [0,T]} \|X^R(t \wedge \tau^R_M)\|_{H^2}^{2p} \right] + c \mathbb{E}\left[ \int_0^{T \wedge \tau^R_M} 1 + \|X^R(\delta)\|_{H^2}^{2p} d\delta \right] \]

by Burkholder-Davis-Gundy inequality, (18a) and Young’s inequality. Applying (63) - (66) in (62) and subtracting \( \frac{1}{4} \mathbb{E}[\sup_{t \in [0,T]} \|X^R(t \wedge \tau^R_M)\|_{H^2}^{2p}] \) from both sides give

\[ \mathbb{E}\left[ \sup_{t \in [0,T]} \|X^R(t \wedge \tau^R_M)\|_{H^2}^{2p} \right] + 2\eta_p \mathbb{E}\left[ \int_0^{T \wedge \tau^R_M} \|\nabla b^R\|_{H^2}^2 \|X^R\|_{H^2}^{2p-2} d\delta \right] \]

\[ \leq 2 \mathbb{E}\left[ \|X^R(0)\|_{H^2}^{2p} \right] + (c_p + 1)c_N \mathbb{E}\left[ \int_0^T 1 + \sup_{\lambda \in [0,\delta]} \|X^R(\lambda \wedge \tau^R_M)\|_{H^2}^{2p} d\delta \right] \tag{67} \]

from which Gronwall’s inequality and taking \( M \to \infty \) deduce (55b). Finally, we note that (55a) follows from (55b). Let us return to (67), take \( p = 1 \), raise to the power of \( \frac{p}{2} \) and take supremum over \( t \in [0,T] \) on the right and then left sides, as
We may estimate well as expectations $E$ to deduce
\[
E\left( \left( \int_0^{T \wedge \tau^R_M} \| \nabla b^R \|^2_{H^s} \, d\delta \right)^{\frac{p_0}{2}} \right)
\]
by (18a),
\[
E\left( \left( \int_0^{T \wedge \tau^R_M} \| X^R \|^2_{H^s} \, d\delta \right)^{\frac{p_0}{2}} \right) \lesssim p_0 \| X^R(0) \|_{H^{s'}}^p + E\left[ \left( c_N \int_0^{T \wedge \tau^R_M} \| X^R \|^2_{H^s} \, d\delta \right)^{\frac{p_0}{2}} \right]
\]
\[
+ E\left[ \sup_{t \in [0,T]} \left( J^s X^R, S_R J^s \Phi(X^R(\delta)) \right) dW \right]^{\frac{p_0}{2}}
\]
\[
+ E\left[ \sup_{t \in [0,T]} \int_0^{t \wedge \tau^R_M} \int_Z \| J^s X^R + S_R J^s \gamma(X^R(\delta), z) \|^2_{L^2} - \| X^R(\delta) \|^2_{H^s}
\]
\[
- 2(J^s X^R, S_R J^s \gamma(X^R(\delta), z)) \mathcal{N}(d\delta, dz) \right]^{\frac{p_0}{2}}
\]
\[
+ E\left[ \sup_{t \in [0,T]} \int_0^{t \wedge \tau^R_M} \int_Z (J^s X^R, S_R J^s \gamma(X^R(\delta), z)) \mathcal{N}(d\delta, dz) \right]^{\frac{p_0}{2}}.
\]
We may estimate
\[
E\left[ \| X^R(0) \|_{H^{s'}}^p \right] \leq E\left[ \| X_0 \|_{H^{s'}}^{p_0} \right]
\]
by (18a),
\[
E\left[ \left( \int_0^{T \wedge \tau^R_M} \| X^R \|^2_{H^s} \, d\delta \right)^{\frac{p_0}{2}} \right] \lesssim E\left[ \sup_{\delta \in [0,T]} \| X^R(\delta \wedge \tau^R_M) \|_{H^{s'}}^{p_0} \right],
\]
\[
E\left[ \sup_{t \in [0,T]} \int_0^{t \wedge \tau^R_M} \| J^s X^R, S_R J^s \Phi(X^R(\delta)) \|_{H^{s'}} \, dW \right]^{\frac{p_0}{2}} \lesssim E\left[ \sup_{t \in [0,T]} 1 + \| X^R(t \wedge \tau^R_M) \|_{H^{s'}}^{p_0} \right],
\]
due to Burkholder-Davis-Gundy inequality and (11).
\[
E\left[ \sup_{t \in [0,T]} \int_0^{t \wedge \tau^R_M} \int_Z \| J^s X^R + S_R J^s \gamma(X^R(\delta), z) \|^2_{L^2} - \| X^R(\delta) \|^2_{H^s}
\]
\[
- 2(J^s X^R, S_R J^s \gamma(X^R(\delta), z)) \mathcal{N}(d\delta, dz) \right]^{\frac{p_0}{2}}
\]
\[
+ E\left[ \sup_{t \in [0,T]} \int_0^{t \wedge \tau^R_M} \int_Z (J^s X^R, S_R J^s \gamma(X^R(\delta), z)) \mathcal{N}(d\delta, dz) \right]^{\frac{p_0}{2}}
\]
by (11), as well as
\[
E\left[ \sup_{t \in [0,T]} \int_0^{t \wedge \tau^R_M} \int_Z (J^s X^R, S_R J^s \gamma(X^R(\delta), z)) \mathcal{N}(d\delta, dz) \right]^{\frac{p_0}{2}}
\]
\[
\lesssim E\left[ \sup_{t \in [0,T]} 1 + \| X^R(t \wedge \tau^R_M) \|_{H^{s'}}^{p_0} \right],
\]
by Burkholder-Davis-Gundy inequality and (11). Thus, using the hypothesis that $X_0 \in L^{p_0}(\Omega; H^{s'}(\mathbb{R}^3))$ and (55b), after taking $M \to \infty$ we may conclude that indeed, (55c) holds. This completes the proof of Proposition 4.3.

Let us define for the rest of the manuscript
\[
\tau_M \triangleq \liminf_{R \to \infty} \tau^R_M
\]
with $\tau^R_M = \inf\{ t \geq 0 : \| X^R(t) \|_{H^s} \geq M \}$, and $X^R$ is the solution to the truncated Hall-MHD system with the cutoff (22a)-(22b), which is known to be unique due
to Proposition 4.1, Proposition 4.2 and Proposition 4.3. We emphasize that \( \tau_M \) does not depend on \( R \), and we also know from Proposition 4.3 that \( \tau_M > 0 \) with probability one, as in particular, the upper bound in (55b) is independent of \( R \).

**Proposition 4.4.** Under the hypothesis of Theorem 2.1, for any \( T > 0 \) fixed, \( M \in \mathbb{R}_+ \), \( R_2 \geq R_1 \) such that \( (X^{R_1}, T \wedge \tau_{R_1}^M) \) and \( (X^{R_2}, T \wedge \tau_{R_2}^M) \) solve the truncated Hall-MHD system with the cutoff \((22a)-(22b)\) with the corresponding \( R_i, i \in \{1, 2\} \), the following equality holds:

\[
\lim_{R_1 \to \infty} \sup_{R_2 \geq R_1} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_M]} \|X^{R_1} - X^{R_2}(t)\|_{H^s}^2 \right] + 2\eta \mathbb{E} \left[ \int_0^{T \wedge \tau_M} \|\nabla(b^{R_1} - b^{R_2})\|^2_{H^s} d\delta \right] = 0,
\]

where \( \iota \in \left(\frac{5}{2}, s - 1\right) \), \( \tau_{R_i}^M \triangleq \inf \{ t \geq 0 : \|X^{R_i}(t)\|_{H^s} \geq M \} \) and \( \tau_M \) is defined in (68).

It turns out that the proof of Proposition 4.4 is quite difficult and nontrivial. The issue, in contrast to [36, Proposition 3.1], is that we have appropriately included the cutoff functions. Because the only useful upper bound for \( |\psi_N(k^{R_1}) - \psi_N(k^{R_2})| \) is \( \|X^{R_1} - X^{R_2}\|_{H^s} \), a standard approach of proving such a Cauchy property in \( L^2 \) (e.g. [31, Lemma 3.7]) will not work (see (71). Consequently we must prove the Cauchy property in \( H^s \); the latter result is stronger because being Cauchy in \( H^s \) implies the same in \( L^2 \). The fact that we must prove the Cauchy property in \( H^s \) for \( \iota > 0 \) instead of \( L^2 \) brings about a surprising difficulty; the following terms

\[ XVIII_4, XVIII_8 + XVIII_{16}, XVIII_{12}, XVIII_{19} \]

in (70) would all be zero due to (1c) and (16c) if \( \iota = 0 \) but not otherwise. For example,

\[
- \psi_N(k^{R_2}) \int S_{R_2} u^{R_2} \cdot \nabla(u^{R_1} - u^{R_2}) \cdot (u^{R_1} - u^{R_2}) = 0,
\]

\[
- \psi_N(k^{R_2}) \int J^i S_{R_2} u^{R_2} \cdot \nabla(u^{R_1} - u^{R_2}) \cdot J^i (u^{R_1} - u^{R_2}) \neq 0.
\]

If one goes through the proof of [31, Lemma 3.7] carefully, it is crucial that such terms vanish. We overcome this difficulty by effectively taking advantage of the stopping time and discovering the appropriate range of \( \iota \in \left(\frac{5}{2}, s - 1\right) \); indeed, the range of \( \iota \) in particular will play a major role throughout the proof of Proposition 4.4.

**Proof of Proposition 4.4.** We compute the equation of \( d(X^{R_1} - X^{R_2}) \), apply \( J^i \), apply Itô’s formula (e.g. [39, Theorem 5.1], [1, Theorem 4.4.7]) with \( f(t, x) = x^2 \)
and integrate over $\mathbb{R}^3$ to obtain

\[
\|(X^{R_1} - X^{R_2})(t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2})\|_{H^s}^2 + 2\eta \int_{0}^{t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2}} \| \nabla (b^{R_1} - b^{R_2}) \|_{H^s}^2 \, d\delta \\
= \|(X^{R_1} - X^{R_2})(0)\|_{H^s}^2 + 2 \int_{0}^{t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2}} (J'(X^{R_1} - X^{R_2}), J'(\psi_N(k^{R_1}))S_{R_1}[-B(X^{R_1}) - HX^{R_1}] \\
- J'(\psi_N(k^{R_2}))S_{R_2}[-B(X^{R_2}) - HX^{R_2}]) \, d\delta \\
+ \int_{0}^{t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2}} \| S_{R_1} \Phi(X^{R_1}) - S_{R_2} \Phi(X^{R_2}) \|_{H^s}^2 \, d\delta \\
+ 2 \int_{0}^{t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2}} (J'(X^{R_1} - X^{R_2}), J'[S_{R_1} \Phi(X^{R_1}) - S_{R_2} \Phi(X^{R_2})]) \, dW \\
+ 2 \int_{0}^{t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2}} (J'(X^{R_1} - X^{R_2}), J'[S_{R_1} \gamma(X^{R_1}) - S_{R_2} \gamma(X^{R_2})]) \tilde{N}(d\delta, dz) \\
+ \int_{0}^{t \wedge \tau_{M}^{R_1} \wedge \tau_{M}^{R_2}} \| S_{R_1} \gamma(X^{R_1}(\delta)) - S_{R_2} \gamma(X^{R_2}(\delta)) \|_{H^s}^2 \, d\delta, dz, \tag{69}
\]
Within the first integral, let us first rewrite

\[ X^{VIII} \triangleq (J^i(X^{R_1} - X^{R_2}), J^i\psi_N(k^{R_1})S_{R_1}[-B(X^{R_1}) - HX^{R_1}] - J^i\psi_N(k^{R_2})S_{R_2}[-B(X^{R_2}) - HX^{R_2}]) \]

\[ = -[\psi_N(k^{R_1}) - \psi_N(k^{R_2})] \int J^iS_{R_1}(u^{R_1} \cdot \nabla)u^{R_1} \cdot J^i(u^{R_1} - u^{R_2}) - \psi_N(k^{R_2}) \int J^i(S_{R_1} - S_{R_2})(u^{R_1} \cdot \nabla)u^{R_1} \cdot J^i(u^{R_1} - u^{R_2}) - \psi_N(k^{R_2}) \int J^iS_{R_2}(u^{R_1} - u^{R_2}) \cdot \nabla u^{R_1} \cdot J^i(u^{R_1} - u^{R_2}) - \psi_N(k^{R_2}) \int J^iS_{R_2}u^{R_2} \cdot \nabla(u^{R_1} - u^{R_2}) \cdot J^i(u^{R_1} - u^{R_2}) + [\psi_N(k^{R_1}) - \psi_N(k^{R_2})] \int J^iS_{R_1}(b^{R_1} \cdot \nabla)b^{R_1} \cdot J^i(b^{R_1} - b^{R_2}) + \psi_N(k^{R_2}) \int J^i(S_{R_1} - S_{R_2})(b^{R_1} \cdot \nabla)b^{R_1} \cdot J^i(b^{R_1} - b^{R_2}) + \psi_N(k^{R_2}) \int J^iS_{R_2}(b^{R_1} - b^{R_2}) \cdot \nabla b^{R_1} \cdot J^i(b^{R_1} - b^{R_2}) + \psi_N(k^{R_2}) \int J^iS_{R_2}b^{R_2} \cdot \nabla(b^{R_1} - b^{R_2}) \cdot J^i(b^{R_1} - b^{R_2}) - [\psi_N(k^{R_1}) - \psi_N(k^{R_2})] \int J^iS_{R_1}(u^{R_1} \cdot \nabla)u^{R_1} \cdot J^i(b^{R_1} - b^{R_2}) + \psi_N(k^{R_2}) \int J^i(S_{R_1} - S_{R_2})(b^{R_1} \cdot \nabla)u^{R_1} \cdot J^i(b^{R_1} - b^{R_2}) + \psi_N(k^{R_2}) \int J^iS_{R_2}(b^{R_1} - b^{R_2}) \cdot \nabla u^{R_1} \cdot J^i(b^{R_1} - b^{R_2}) + \psi_N(k^{R_2}) \int J^iS_{R_2}b^{R_2} \cdot \nabla(u^{R_1} - u^{R_2}) \cdot J^i(b^{R_1} - b^{R_2}) - [\psi_N(k^{R_1}) - \psi_N(k^{R_2})] \int J^iS_{R_2}\nabla \times (j^{R_1} \times b^{R_1}) \cdot J^i(b^{R_1} - b^{R_2}) - \psi_N(k^{R_2}) \int J^i(S_{R_1} - S_{R_2})\nabla \times (j^{R_1} \times b^{R_1}) \cdot J^i(b^{R_1} - b^{R_2}) - \psi_N(k^{R_2}) \int J^iS_{R_2}\nabla \times ((j^{R_1} - j^{R_2}) \times b^{R_1}) \cdot J^i(b^{R_1} - b^{R_2}) - \psi_N(k^{R_2}) \int J^iS_{R_2}\nabla \times (j^{R_2} \times (b^{R_1} - b^{R_2})) \cdot J^i(b^{R_1} - b^{R_2}) \]

\[ \triangleq \sum_{i=1}^{20} X^{VIII}_i. \]
Now let us estimate

$$
\sum_{i \in \{1,5,9,13\}} \text{XVIII}_i \\
\leq |\psi_N(kR_1) - \psi_N(kR_2)| \\
\times \left[ \|[(u^{R_1} \cdot \nabla) u^{R_1}]H\|H \cdot \|u^{R_1} - u^{R_2}\|H + \|(b^{R_1} \cdot \nabla) b^{R_1}\|H \cdot \|u^{R_1} - u^{R_2}\|H \right. \\
+ \left. \|j^{R_1} \times b^{R_1}\|H \cdot \|\nabla (b^{R_1} - b^{R_2})\|H \right] \\
\lesssim \|X^{R_1} - X^{R_2}\|H \cdot \|X^{R_1}\|^2 H \cdot \|X^{R_1} - X^{R_2}\|H + \|X^{R_1}\|_H^2 \cdot \|\nabla (b^{R_1} - b^{R_2})\|H \\
\lesssim \frac{\eta}{8} \|\nabla (b^{R_1} - b^{R_2})\|_H^2 + c \|X^{R_1} - X^{R_2}\|_H^2 \left[ \|X^{R_1}\|_H + \|X^{R_1} - X^{R_2}\|_H \right]
$$

by (70), integration by parts, Hölder’s inequality, (18a), that $|\psi_N(kR_1) - \psi_N(kR_2)| \leq \|X^{R_1} - X^{R_2}\|_H$, similarly to (27), that $\iota \in (\frac{5}{2}, s-1)$ and Young’s inequality. Next, we estimate

$$
\sum_{i \in \{2,6,10,14\}} \text{XVIII}_i \lesssim \|(S_{R_1} - S_{R_2}) \text{div}(u^{R_1} \otimes u^{R_1})\|H \cdot \|u^{R_1} - u^{R_2}\|_H, \\
+ \|(S_{R_1} - S_{R_2}) \text{div}(b^{R_1} \otimes b^{R_1})\|H \cdot \|u^{R_1} - u^{R_2}\|_H, \\
+ \|(S_{R_1} - S_{R_2}) \text{div}(u^{R_1} \otimes b^{R_1})\|H \cdot \|b^{R_1} - b^{R_2}\|_H, \\
+ \|(S_{R_1} - S_{R_2}) \text{div}(b^{R_1} \otimes u^{R_1})\|H \cdot \|b^{R_1} - b^{R_2}\|_H, \\
\lesssim \frac{1}{R_1^{s-1-\iota}} \|X^{R_1}\|_H^2 \|X^{R_1} - X^{R_2}\|_H, \\
\lesssim \frac{M^4}{R_1^{2(s-1-\iota)}} + \|X^{R_1} - X^{R_2}\|_H^2,
$$

where we used (70), Hölder’s inequalities and (18c). Next, we estimate

$$
\text{XVIII}_{18} = -\psi_N(kR_2) \int (S_{R_1} - S_{R_2}) J^\iota (j^{R_1} \times b^{R_1}) J^\iota \nabla \times (b^{R_1} - b^{R_2}) \\
\leq \psi_N(kR_2) \|(S_{R_1} - S_{R_2}) J^\iota (j^{R_1} \times b^{R_1}) \|L^2 \|\nabla (b^{R_1} - b^{R_2})\|H, \\
\lesssim \frac{1}{R_1^{s-1-\iota}} \|j^{R_1} \times b^{R_1}\|_{H^{s-1}} \|\nabla (b^{R_1} - b^{R_2})\|_H, \\
\lesssim \frac{\eta}{8} \|\nabla (b^{R_1} - b^{R_2})\|_H^2 + \frac{c}{R_1^{2(s-1-\iota)}} \|j^{R_1}\|_{H^{s-1}}^2 \|b^{R_1}\|_{H^{s-1}}^2, \\
\lesssim \frac{\eta}{8} \|\nabla (b^{R_1} - b^{R_2})\|_H^2 + \frac{cM^4}{R_1^{2(s-1-\iota)}},
$$

by (73), integration by parts, Hölder’s inequality, (18a), that $|\psi_N(kR_1) - \psi_N(kR_2)| \leq \|X^{R_1} - X^{R_2}\|_H$, similarly to (27), that $\iota \in (\frac{5}{2}, s-1)$ and Young’s inequality.
where we used \((70)\), Hölder’s inequality, \((18a)\), Young’s inequality and that \(\iota \in \left(\frac{s}{2}, s - 1\right)\). Next, we estimate

\[
\sum_{i \in \{3, 7, 11, 15\}} \text{ XVIII}_i
\]

\[
\lesssim \|(u^{R_1} - u^{R_2}) \cdot \nabla u^{R_1}\|_{H^s} \|u^{R_1} - u^{R_2}\|_{H^s}
+ \|(b^{R_1} - b^{R_2}) \cdot \nabla b^{R_1}\|_{H^s} \|u^{R_1} - u^{R_2}\|_{H^s}
+ \|(u^{R_1} - u^{R_2}) \cdot \nabla b^{R_1}\|_{H^s} \|b^{R_1} - b^{R_2}\|_{H^s}
+ \|(b^{R_1} - b^{R_2}) \cdot \nabla u^{R_1}\|_{H^s} \|b^{R_1} - b^{R_2}\|_{H^s}
\]

\[
\lesssim \|u^{R_1} - u^{R_2}\|_{H^s} \|\nabla u^{R_1}\|_{H^s} + \|b^{R_1} - b^{R_2}\|_{H^s} \|\nabla b^{R_1}\|_{H^s} \|u^{R_1} - u^{R_2}\|_{H^s}
+ \|u^{R_1} - u^{R_2}\|_{H^s} \|\nabla b^{R_1}\|_{H^s} \|b^{R_1} - b^{R_2}\|_{H^s}
+ \|b^{R_1} - b^{R_2}\|_{H^s} \|\nabla u^{R_1}\|_{H^s}
\]

\[
\lesssim \|X^{R_1} - X^{R_2}\|_{H^s}^2 M
\]

where we used \((70)\), Hölder’s inequality, \((18a)\) and that \(\iota \in \left(\frac{s}{2}, s - 1\right)\). Next, we estimate

\[
\text{ XVIII}_{19}
\]

\[
= - \psi_N(k^{R_2}) \int J^j((j^{R_1} - j^{R_2}) \times b^{R_1}) \cdot J^j(j^{R_1} - j^{R_2})
\]

\[
= - \psi_N(k^{R_2}) \int (J^j((j^{R_1} - j^{R_2}) \times b^{R_1})
- [J^j(j^{R_1} - j^{R_2}) \times b^{R_1}]) \cdot J^j(j^{R_1} - j^{R_2})
\]

\[
\lesssim (\|j^{R_1} - j^{R_2}\|_{H^{s-1}} \|\nabla b^{R_1}\|_{L^\infty} + \|j^{R_1} - j^{R_2}\|_{L^\infty} \|b^{R_1}\|_{H^{s-1}}) \|j^{R_1} - j^{R_2}\|_{H^s}
\]

\[
\lesssim (\|b^{R_1} - b^{R_2}\|_{H^s} \|b^{R_1}\|_{H^s} + \|b^{R_1} - b^{R_2}\|_{H^s} \|b^{R_1}\|_{H^s}) \|\nabla (b^{R_1} - b^{R_2})\|_{H^s}
\]

\[
\lesssim \frac{\eta}{8} \|\nabla (b^{R_1} - b^{R_2})\|_{H^s}^2 + cM^2 \|X^{R_1} - X^{R_2}\|_{H^s}^2
\]

where we used \((70)\), \((16c)\), Lemma 3.1, and Young’s inequality. Next, we estimate

\[
\sum_{i \in \{4, 8, 12, 16\}} \text{ XVIII}_i
\]

\[
= - \psi_N(k^{R_2}) \int [J^j[u^{R_2} \cdot \nabla (u^{R_1} - u^{R_2})] - u^{R_2} \cdot \nabla J^j(u^{R_1} - u^{R_2})] \cdot J^j(u^{R_1} - u^{R_2})
\]

\[
+ \psi_N(k^{R_2}) \int [J^j[b^{R_2} \cdot \nabla (b^{R_1} - b^{R_2})] - b^{R_2} \cdot \nabla J^j(b^{R_1} - b^{R_2})] \cdot J^j(u^{R_1} - u^{R_2})
\]

\[
- \psi_N(k^{R_2}) \int [J^j[u^{R_2} \cdot \nabla (b^{R_1} - b^{R_2})] - u^{R_2} \cdot \nabla J^j(b^{R_1} - b^{R_2})] \cdot J^j(b^{R_1} - b^{R_2})
\]

\[
+ \psi_N(k^{R_2}) \int [J^j[b^{R_2} \cdot \nabla (u^{R_1} - u^{R_2})] - b^{R_2} \cdot \nabla J^j(u^{R_1} - u^{R_2})] \cdot J^j(b^{R_1} - b^{R_2})
\]

\[
\lesssim \|\nabla X^{R_2}\|_{L^\infty} \|X^{R_1} - X^{R_2}\|_{H^s} + \|X^{R_2}\|_{H^s} \|\nabla (X^{R_1} - X^{R_2})\|_{L^\infty} \|X^{R_1} - X^{R_2}\|_{H^s}
\]

\[
\lesssim M \|X^{R_1} - X^{R_2}\|_{H^s}^2
\]
due to (70), Hölder’s inequality and Lemma 3.1. Finally, we estimate

\[ X_{20} = -\psi_N(R)^r \int J^t(R) \times (bR_1 - bR_2) \cdot J^t \nabla \times (bR_1 - bR_2) \]

\[ \leq \| J^t(R) \times (bR_1 - bR_2) \|_{H^s} \| \nabla (bR_1 - bR_2) \|_{H^t} \]

\[ \leq \| J^t(R) \|_{H^s} (\| bR_1 - bR_2 \|_{H^s} \| \nabla (bR_1 - bR_2) \|_{H^t}) \]

\[ \leq \frac{\eta}{8} \| \nabla (bR_1 - bR_2) \|_{H^t}^2 + cM^2 \| bR_1 - bR_2 \|_{H^t}^2 \]  \tag{77}

where we used integration by parts, Hölder’s inequality and Young’s inequality. Thus, applying (71)-(77) to (70) gives

\[ X_{18} \leq \frac{\eta}{2} \| \nabla (bR_1 - bR_2) \|_{H^t}^2 \]

\[ + c(M^2 + M^4 + M + 1)(X_{R_1} - X_{R_2})^2 + \frac{cM^4}{R_1^{2(s-1-c)}} \]  \tag{78}

Applying (78) to (69), subtracting \( \eta \| \nabla (bR_1 - bR_2) \|_{H^t}^2 \) from both sides, taking supremum over \( t \in [0, T] \) on the right and then left sides, and then expectation \( E \) lead to

\[ E[ \sup_{t \in [0, T]} \| (X_{R_1} - X_{R_2})(t \wedge \tau_{R_1} \wedge \tau_{R_2}) \|_{H^t}^2 ] \]

\[ + \eta E \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| \nabla (bR_1 - bR_2) \|_{H^t}^2 \, d\delta \]

\[ \leq E[ \| (X_{R_1} - X_{R_2})(0) \|_{H^t}^2 ] \]

\[ + c(M^2 + M^4 + M + 1)E \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| X_{R_1} - X_{R_2} \|_{H^t}^2 \, d\delta \]

\[ + \frac{cM^4}{R_1^{2(s-1-c)}} (T \wedge \tau_{R_1} \wedge \tau_{R_2}) \]

\[ + E \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| S_{R_1} \Phi(X_{R_1}) - S_{R_2} \Phi(X_{R_2}) \|_{H^t}^2 \, d\delta \]

\[ + 2E[ \sup_{t \in [0, T]} \| J^t(X_{R_1} - X_{R_2}) \|_{H^t}^2 ] \]

\[ + 2E[ \sup_{t \in [0, T]} \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| J^t(X_{R_1} - X_{R_2}) \|_{H^t}^2 ] \]

\[ + 2E[ \sup_{t \in [0, T]} \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| S_{R_1} \gamma(X_{R_1}) - S_{R_2} \gamma(X_{R_2}) \|_{H^t}^2 ] \]

\[ + 2E[ \sup_{t \in [0, T]} \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| S_{R_1} \gamma(X_{R_1}) - S_{R_2} \gamma(X_{R_2}) \|_{H^t}^2 ] \]

\[ + 2E[ \sup_{t \in [0, T]} \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| S_{R_1} \gamma(X_{R_1}(\delta)) - S_{R_2} \gamma(X_{R_2}(\delta)) \|_{H^t}^2 ] \]

\[ + 2E[ \sup_{t \in [0, T]} \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| S_{R_1} \gamma(X_{R_1}(\delta)) - S_{R_2} \gamma(X_{R_2}(\delta)) \|_{H^t}^2 ] \]

\[ + 2E[ \sup_{t \in [0, T]} \int_0^{T \wedge \tau_{R_1} \wedge \tau_{R_2}} \| S_{R_1} \gamma(X_{R_1}(\delta)) - S_{R_2} \gamma(X_{R_2}(\delta)) \|_{H^t}^2 ] \]
where we now estimate the last four terms. Firstly, from (79),

\[
E\left[ \int_0^{T^\wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} \| S_{R_1} \Phi(X_{R_1}) - S_{R_2} \Phi(X_{R_2}) \|_{H^s}^2 d\delta \right] \\
\lesssim \frac{1}{R_1^{\eta-1}} E\left[ \int_0^{T^\wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} \| \Phi(X_{R_1}) \|_{H^s}^2 d\delta \right] \\
+ E\left[ \int_0^{T^\wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} \| \Phi(X_{R_1}) - \Phi(X_{R_2}) \|_{H^s}^2 d\delta \right]
\]

\[
\lesssim \left( \frac{1 + M^2}{R_1^{2(s-\eta)}} \right) (T \wedge \tau_{R_1}^M \wedge \tau_{R_2}^M) + E\left[ \int_0^{T^\wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} \| X_{R_1} - X_{R_2} \|_{H^s}^2 d\delta \right] 
\]

by Minkowski’s inequality, (18c), (18a), (11) and (12). Secondly, from (79),

\[
2E\left[ \sup_{t \in [0,T]} \left| \int_0^{t \wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} J'(X_{R_1} - X_{R_2}), d\delta \right| \right] \\
\lesssim E\left[ \int_0^{T^\wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} \| X_{R_1} - X_{R_2} \|_{H^s}^2 d\delta \right] \\
\times \left( \| S_{R_1} - S_{R_2} \|_2 \right)^2 + \| S_{R_2} \|_2 \left( \left\| \Phi(X_{R_1}) - \Phi(X_{R_2}) \right\|_2 \right)^2 d\delta \right) \]

\[
\lesssim \frac{M + M^2}{R_1^{2s-1}} (T \wedge \tau_{R_1}^M \wedge \tau_{R_2}^M)^{\frac{1}{2}} \\
+ E\left[ \sup_{t \in [0,T]} \left| \left( X_{R_1} - X_{R_2} \right)(t \wedge \tau_{R_1}^M \wedge \tau_{R_2}^M) \right|_{H^s}^2 d\delta \right] \]

\[
\lesssim \frac{M + M^2}{R_1^{2s-1}} (T \wedge \tau_{R_1}^M \wedge \tau_{R_2}^M)^{\frac{1}{2}} + \frac{c(M + M^2)(T \wedge \tau_{R_1}^M \wedge \tau_{R_2}^M)^{\frac{1}{2}}}{R_1^{s-1}} + cE\left[ \int_0^{T^\wedge \tau_{R_1}^M \wedge \tau_{R_2}^M} \| X_{R_1} - X_{R_2} \|_{H^s}^2 d\delta \right] 
\]
by Burkholder-Davis-Gundy inequality (e.g. [21, Proposition 2.4 (i)], [25, pg. 166]), (18c), (18a), (11), (12), Young’s inequality. Thirdly, from (79),

\[ \leq \]

by Burkholder-Davis-Gundy inequality (e.g. [21, Proposition 2.4 (i)], [25, pg. 166]), (18c), (18a), (11), (12), and Young’s inequality. Finally, from (79),

\[ \leq \]

by Burkholder-Davis-Gundy inequality (e.g. [21, Proposition 2.4 (i)], [25, pg. 166]), (18c), (18a), (11), (12) and Young’s inequality.
where we used (18c), (18a), (11) and (12). Applying (80) - (83) to (79) and subtracting $\frac{1}{2}\mathbb{E}[\sup_{t \in [0,T]} \|X^{R_1} - X^{R_2}(t \wedge \tau_M^{R_1} \wedge \tau_M^{R_2})\|_{H^1}^2]$ from both sides give

$$
\mathbb{E}[\sup_{t \in [0,T]} \|X^{R_1} - X^{R_2}(t \wedge \tau_M^{R_1} \wedge \tau_M^{R_2})\|_{H^1}^2] \\
+ 2\eta \mathbb{E}\left[\int_0^{T \wedge \tau_M^{R_1} \wedge \tau_M^{R_2}} \|\nabla(b^{R_1} - b^{R_2})\|_{H^1}^2 \, d\delta\right] \\
\lesssim \frac{\mathbb{E}[\|X_0\|_{H^{\frac{7}{2}}}^2]}{R_1^{2(s-1)}} + [M^2 + M^4 + M + 1] \mathbb{E}\left[\int_0^{T \wedge \tau_M^{R_1} \wedge \tau_M^{R_2}} \|X^{R_1} - X^{R_2}\|_{H^1}^2 \, d\delta\right] \\
+ \left(\frac{M^4}{R_1^{2(s-1)}} + \frac{1 + M^2}{R_1^{2(s-1)}}\right) (T \wedge \tau_M^{R_1} \wedge \tau_M^{R_2}) + \frac{M + M^2}{R_1^{2(s-1)}} (T \wedge \tau_M^{R_1} \wedge \tau_M^{R_2})^{\frac{1}{2}}
$$

where we also used the estimate of

$$\mathbb{E}[\|X^{R_1} - X^{R_2}(0)\|_{H^{\frac{7}{2}}}^2] = \mathbb{E}[\|(S_{R_1} - S_{R_2})X(0)\|_{H^{\frac{7}{2}}}^2] \lesssim \frac{1}{R_1^{2(s-1)}} \mathbb{E}[\|X(0)\|_{H^{\frac{7}{2}}}^2].$$

Gronwall’s inequality type argument leads to the desired result, as we note that $\tau_M \leq \lim_{R_1 \to \infty} (\tau_M^{R_1} \wedge \tau_M^{R_2})$ due to (68). This completes the proof of Proposition 4.4.

4.1. Global existence of a solution to the Hall-MHD system with the cutoff. By Proposition 4.4 and property of Cauchy sequence, we are able to deduce

$$X^R(\cdot \wedge \tau_M) \to X(\cdot \wedge \tau_M) \text{ strongly in } L^2(\Omega; L^\infty([0,T]; H^s(\mathbb{R}^3))),$$

$$b^R(\cdot \wedge \tau_M) \to b(\cdot \wedge \tau_M) \text{ strongly in } L^2(\Omega; L^2([0,T]; H^{s+1}(\mathbb{R}^3))),$$

as $R \to \infty$, where $\tau_M$ is defined in (68). Since similarly to $X^R$, it is clear that $X$ also satisfies the previous estimates of $L^{p_0}(\Omega; L^\infty(0,T; H^s(\mathbb{R}^3)))$ for any $p_0 \in [4, \infty)$ in Proposition 4.3 (although it can be given here, we give details of this estimate in the proof of Theorem 2.2). Sobolev interpolation theorem and (85a) - (85b) together imply

$$X^R(\cdot \wedge \tau_M) \to X(\cdot \wedge \tau_M) \text{ strongly in } L^2(\Omega; L^\infty([0,T]; H^{s'}(\mathbb{R}^3))),$$

$$b^R(\cdot \wedge \tau_M) \to b(\cdot \wedge \tau_M) \text{ strongly in } L^2(\Omega; L^2([0,T]; H^{s'+1}(\mathbb{R}^3))),$$

for any $s' \leq s$; we note that because $s > \frac{7}{2}$ by hypothesis, we may still choose $s' > \frac{7}{2}$ so that such convergence holds.

We now take the limit from the truncated Hall-MHD system with the cutoff (22a) - (22b) to the Hall-MHD system with the cutoff (21). Firstly for the diffusive term,

$$\|(AX^R - AX)(\cdot \wedge \tau_M)\|_{L^1(\Omega; L^2([0,T]; H^{s'-1}))} \lesssim \mathbb{E}\left[\int_0^T \|(b^R - b)(\cdot \wedge \tau_M)\|_{H^{s'+1}}^2 \, d\delta\right]^{\frac{1}{2}} \to 0
$$

(87)
as $R \to \infty$ by (86b). Next, we work on the nonlinear terms:

$$\|(\psi_N(k^R)S_RB(X^R) - \psi_N(k)B(X))(\cdot \wedge \tau_M)\|_{L^1(\Omega;L^\infty([0,T];H^{s'-1}))}$$

\[ \lesssim \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} |\psi_N(k^R) - \psi_N(k)| \times \|S_R[(u^R \cdot \nabla)u^R - (b^R \cdot \nabla)b^R + (u^R \cdot \nabla)b^R - (b^R \cdot \nabla)u^R]\|_{H^{s'-1}} + \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|\psi_N(k)(S_R - Id)\|_{H^{s'-1}} \times [(u^R \cdot \nabla)u^R - (b^R \cdot \nabla)b^R + (u^R \cdot \nabla)b^R - (b^R \cdot \nabla)u^R]\|_{H^{s'-1}} \]

\[ + \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|\psi_N(k)(S_R - Id)\|_{H^{s'-1}} \times [(u^R \cdot \nabla)u^R - (b^R \cdot \nabla)b^R + (u^R \cdot \nabla)b^R - (b^R \cdot \nabla)u^R]\|_{H^{s'-1}} \]

Finally, we estimate $XIX + XX + XXX$ where we estimate all three terms as follows. Firstly,

$$XIX \lesssim \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|X^R - X\|_{H^s} \left( \sup_{t \in [0,T \wedge \tau_M]} \|u^R\|_{H^{s'}}^2 + \|b^R\|_{H^{s'}}^2 + \|u^R\|_{H^{s'}} \|b^R\|_{H^{s'}} \right)$$

\[ \lesssim \left( \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|(X^R - X)(t)\|_{L^2}^2 \right)^{\frac{1}{2}} \to 0 \]  

(89)

as $R \to \infty$ where we used (88), an estimate similar to (27), that $s' > \frac{7}{2}, s' < s$, (55b) and (86a). Secondly, we estimate

$$XX \lesssim \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \frac{1}{R^{s-s'}} \|\text{div}[(u^R \otimes u^R) - (b^R \otimes b^R) + (u^R \otimes b^R) - (b^R \otimes u^R)]\|_{H^{s-1}} \lesssim \frac{1}{R^{s-s'}} \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|X^R(t)\|_{H^{s'}} \to 0$$

(90)

as $R \to \infty$, by (88), (18b), and the hypothesis that $s > \frac{7}{2}$. Finally, we estimate

$$XXI \lesssim \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|X^R - X\|_{H^{s'}} \|X^R\|_{H^{s'}}$$

\[ \lesssim \left( \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|X^R - X\|_{H^{s'}}^2 \right)^{\frac{1}{2}} \left( \mathbb{E}\sup_{t \in [0,T \wedge \tau_M]} \|X^R\|_{H^{s'}}^2 \right)^{\frac{1}{2}} \to 0 \]  

(91)
as $R \to \infty$ where we used \([88]\), Hölder’s inequality, that $s' - 1 > \frac{5}{2}$, \((55b)\) and \((86a)\). Next, we consider the most difficult Hall term:

\[
\| (\psi_N(k) R H X^R - \psi_N(k) H X) (\cdot \wedge \tau_M) \|_{L^1(\Omega; L^2([0,T]; H^{s' - 1}))} \\
\leq \| \psi_N(k) R H X^R - \psi_N(k) H X \|_{L^1(\Omega; L^2([0,T]; H^{s' - 1}))} \\
+ \| \psi_N(k) [H X^R - H X] \|_{L^1(\Omega; L^2([0,T]; H^{s' - 1}))} \\
\leq E \left( \sup_{t \in [0,T]} \| (X^R - X)(t) \|_{H^s} \right)^{\frac{1}{2}} \left( E \left[ \sup_{t \in [0,T]} \| b R(t) \|_{H^{s'}}^2 \right] \right)^{\frac{1}{2}} \\
\times \left( E \left[ \left( \int_0^{T \wedge \tau_M} \| b R(t) \|_{H^{s'+1}}^2 \right) \right] \right)^{\frac{1}{2}} \\
+ \left( E \left[ \sup_{t \in [0,T]} \| b R(t) \|_{H^{s'}}^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \left( \int_0^{T \wedge \tau_M} \| b R - b \|_{H^{s'+1}}^2 \right) \right] \right)^{\frac{1}{2}} \\
+ \left( E \left[ \sup_{t \in [0,T]} \| (b R - b)(t) \|_{H^{s'}}^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \left( \int_0^{T \wedge \tau_M} \| b \|_{H^{s'+1}}^2 \right) \right] \right)^{\frac{1}{2}} \to 0
\]

as $R \to \infty$ by the estimate similar to \((27)\), that $s' > \frac{7}{2}$, Hölder’s inequality, \([86a]\), \((86b)\) and Proposition 4.3. Next, it is immediate that

\[
E \| S R X_0 - X_0 \|_{H^{s'-1}}^2 \lesssim \frac{1}{R} E \| X_0 \|_{H^{s'}}^2 \to 0
\]

as $R \to \infty$ where we used \((18b)\). Next,

\[
\| S_R \left( \int_0^{T \wedge \tau_M} \Phi(X^R) dW(\delta) \right) - \int_0^{T \wedge \tau_M} \Phi(X) dW(\delta) \|_{L^2(\Omega; L^\infty([0,T]; H^{s' - 1}))} \\
\leq E \left( \sup_{t \in [0,T]} \| (S_R - Id) \int_0^{T \wedge \tau_M} \Phi(X^R(\delta)) dW(\delta) \|_{H^{s'-1}}^2 \right) \\
+ E \left( \sup_{t \in [0,T]} \| \int_0^{T \wedge \tau_M} \Phi(X^R(\delta)) - \Phi(X(\delta)) dW(\delta) \|_{H^{s'-1}}^2 \right) \\
\lesssim \frac{1}{R^2} \left( E \left[ \sup_{t \in [0,T]} \left( \int_0^{T \wedge \tau_M} J^{s'} \Phi(X^R(\delta)) dW(\delta) \right)^2 dx \right] \right) \\
+ \left( E \left[ \sup_{t \in [0,T]} \left( \int_0^{T \wedge \tau_M} J^{s'-1} [\Phi(X^R(\delta)) - \Phi(X(\delta))] dW(\delta) \right)^2 dx \right] \right) \\
\lesssim \frac{1}{R^2} E \left( \int_0^{T \wedge \tau_M} 1 + \| X^R \|_{H^{s'}}^2 d\delta \right) + E \left( \int_0^{T \wedge \tau_M} \| X^R - X \|_{H^{s'-1}}^2 d\delta \right) \to 0
\]
as $R \to \infty$ where we used \((18\text{b})\), Burkholder-Davis-Gundy inequality, Young’s inequality, \([8\text{a}]. \text{[11]} \text{ and } \text{[12]}. \) Finally,
\[
\| \int_0^{T \wedge M} \int_{\mathbb{R}^2} S_R \gamma(X^R(\delta-), z) \mathcal{N}(d\delta, dz) - \int_0^{T \wedge M} \int_{\mathbb{R}^2} \gamma(X(\delta-), z) \mathcal{N}(d\delta, dz) \|_{L^2(\Omega; L^\infty([0, T]; H^s)}^2
\]
\[
\leq E \sup_{t \in [0, T]} \| (S_R - Id) \left( \int_0^{T \wedge M} \int_{\mathbb{R}^2} \gamma(X^R(\delta-), z) \mathcal{N}(d\delta, dz) \right) \|_{H^{s-1}}^2
\]
\[
+ E \sup_{t \in [0, T]} \| \int_0^{T \wedge M} \int_{\mathbb{R}^2} \left( \gamma(X^R(\delta-), z) - \gamma(X(\delta-), z) \right) \mathcal{N}(d\delta, dz) \|_{H^{s-1}}^2
\]
\[
\lesssim \frac{1}{R^2} E \int_0^{T \wedge M} \int_{\mathbb{R}^2} \| \gamma(X^R(\delta-), z) \|_{H^{s'}, \lambda}^2 d\delta d\delta
\]
\[
+ E \int_0^{T \wedge M} \int_{\mathbb{R}^2} \| \gamma(X^R(\delta-), z) - \gamma(X(\delta-), z) \|_{H^{s-1}, \lambda}^2 d\delta d\delta
\]
\[
\lesssim \frac{1}{R^2} E \int_0^{T \wedge M} 1 + \| X^R \|_{H^{s'}}^2 d\delta + E \int_0^{T \wedge M} \| X^R - X \|_{H^{s-1}}^2 d\delta \to 0
\]
as $R \to \infty$ by \((18\text{b})\), Burkholder-Davis-Gundy inequality, \([11], [12], \text{ and } [8\text{a}]\). Therefore, due to \((87) - (95)\), we may pass the limits in the system \((22\text{a})-(22\text{b})\) for any $t \in [0, T]$ so that $X$ solves
\[
X(t \wedge \tau_M) = X_0 + \int_0^{t \wedge \tau_M} -AX + \psi_N(k)[-B(X) - HX]d\delta
+ \int_0^{t \wedge \tau_M} \Phi(X(\delta))dW(\delta) + \int_0^{t \wedge \tau_M} \int_{\mathbb{R}^2} \gamma(X(\delta-), z) \mathcal{N}(d\delta, dz)
\]
in $L^1(\Omega; L^2([0, T]; H^{s'-1}(\mathbb{R}^3))), s'> \frac{7}{2}$. Moreover, the right hand side of \((55\text{b})\) in Proposition 4.3 being independent of $M$, by Banach-Alaoglu theorem, we obtain a subsequence, which we just relabel by $X^R(\cdot \wedge \tau_M)$ such that $X^R(\cdot \wedge \tau_M)$ converges to $X(\cdot \wedge \tau_M)$ weak* in $L^{2p}(\Omega; L^\infty([0, T]; H^{s}(\mathbb{R}^3)))$ for all $p \in [1, \frac{7}{2}]$ as $R \to \infty$ so that $X \in L^{2p}(\Omega; L^\infty([0, T]; H^{s}(\mathbb{R}^3)))$. In fact, because the upper bound on \((55\text{b})\) of Proposition 4.3 was also independent of $M$, it is clear that for any fixed $T > 0$, we may now take the limit $M \to \infty$ so that $T \wedge \tau_M \to T$ $\mathbb{P}$-almost surely to deduce the global existence of the solution $X \in L^{2p}(\Omega; L^\infty([0, T]; H^{s}(\mathbb{R}^3)))$ for all $p \in [1, \frac{7}{2}]$ for the Hall-MHD system with the cutoff \([21]\). Finally, from the strong convergence of $X^R \to X$ in $L^2(\Omega; H^{s'}([0, T]; H^{s}(\mathbb{R}^3)))$, it is clear that $X^R$ is $\mathbb{P}$-almost surely convergent uniformly on $[0, T]$ to $X$, so that $X$ is adapted and càdlàg by \([11] \text{ Theorem } 6.2.3\).

4.2. **Path-wise uniqueness of the global solution to the Hall-MHD system with the cutoff.** We now know the global existence of the solution to the Hall-MHD system with the cutoff \([21]\). It is clear from our proof of Proposition 4.1 that an identical proof shows the path-wise uniqueness of the solution to the Hall-MHD system with the cutoff \([21]\). Indeed, in the computations from \((24) \text{ to } (41)\), the properties \((18\text{b}), (18\text{c})\) were never used; we only took advantage of \((18\text{a})\).
4.3. Local existence of a path-wise unique solution to the Hall-MHD system. Let \( c_0 > 0 \) be the constant such that \( \|f\|_{H^s} \leq c_0 \|f\|_{H^r} \) for all \( f \in H^s(\mathbb{R}^3) \). Due to the global existence of the unique solution to the Hall-MHD system with the cutoff function \( (21) \), we know that for each fixed \( N \geq 3, T \in (0, \infty) \), there exist a progressively measurable càdlàg process \( X_N(\cdot) \) such that it satisfies the following Hall-MHD system with the cutoff

\[
X_N(t) = X_N(0) - \int_0^t A X_N(\delta)d\delta + \int_0^t \psi_{c_0 N}(k_N)[-B(X_N) - H X_N]d\delta \\
+ \int_0^t \Phi(X_N(\delta))dW(\delta) + \int_0^t \int_{\mathbb{Z}} \gamma(X_N(\delta), z) \hat{N}(d\delta, dz)
\]

for all \( t \in [0, T \wedge \tau_N] \) where \( \tau_N \triangleq \inf\{t \geq 0 : \|X_N(t)\|_{H^r} \geq N\} \), \( k_N \triangleq \|X_N\|_{H^s} \), \( \ell \in (\frac{5}{3}, s - 1) \). Obviously if \( N_1 < N_2 \), for almost all \( \omega \in \Omega \), \( \tau_{N_1} \leq \tau_{N_2} \) and pathwise uniqueness, \( X_{N_1}(t) = X_{N_2}(t) \) for all \( t \in [0, \tau_{N_1} \wedge \tau_{N_2}] \). We now define

\[
\tau(\omega) \triangleq \lim_{N \to \infty} \tau_N(\omega), \quad X(t) \triangleq \lim_{N \to \infty} X_N(t) \quad \text{for } t \in [0, \tau) \quad \mathbb{P} - \text{a.s.}
\]

so that \((X, \tau)\) is a local strong solution to the Hall-MHD system \( (1a)-(1c) \). In order to prove the pathwise uniqueness, we suppose that \((\tilde{X}, \tilde{\tau})\) is another local strong solution so that by our construction, there must have existed an increasing sequence of stopping times \( \{\tilde{\tau}_N\}_{N \geq 1} \) such that \( \tilde{\tau}_N = \inf\{t \geq 0 : \|\tilde{X}_N(t)\|_{H^r} \geq N\} \), \( \lim_{N \to \infty} \tilde{\tau}_N = \tilde{\tau} \), \((\tilde{X}_N, \tilde{\tau}_N)\) is the strong solution to the Hall-MHD system with the cutoff. But by uniqueness,

\[
X_N(t) = \tilde{X}_N(t) \quad \forall \ t \in [0, \tau_N \wedge \tilde{\tau}_N],
\]

\( \mathbb{P} \)-almost surely for all \( N \geq 1 \) and thus taking \( N \nearrow \infty \), we deduce

\[
X(t) = \tilde{X}(t) \quad \forall \ t \in [0, \tau \wedge \tilde{\tau}]
\]

\( \mathbb{P} \)-almost surely. Now if \( \tau \neq \tilde{\tau} \), then either \( \tau > \tilde{\tau} \) or \( \tilde{\tau} > \tau \). \( \mathbb{P} \)-almost surely. Suppose \( \tilde{\tau} < \tau \). Then

\[
\lim_{t \nearrow \tau} \sup_{\delta \in [0, \delta]} \|\chi(\tau < \tau) X(\delta)\|_{H^r} = \lim_{N \to \infty} \sup_{\delta \in [0, \tau_N]} \|\chi(\tau < \tau) X(\delta)\|_{H^r} = \infty
\]

where we used that \( \lim_{N \to \infty} \tau_N = \tau \), that \( X(t) = \tilde{X}(t) \) for all \( t \in [0, \tau \wedge \tilde{\tau}] \) \( \mathbb{P} \)-almost surely. But this clearly contradicts that \( \tilde{\tau} < \tau \) so that at time \( \tilde{\tau} \), \( \|X(t)\|_{H^r} \) should not blow up. By symmetry, we see that the case \( \tau < \tilde{\tau} \) also leads to a contradiction, and conclude the proof of the path-wise uniqueness of the local solution to the Hall-MHD system \( (1a)-(1c) \).

5. Proof of Theorem 2.2

We fix \( \delta \in (0, 1) \), let \((X, \tau)\) be the local unique strong solution to the Hall-MHD system \( (1a)-(1c) \) so that \( \tau = \lim_{N \to \infty} \tau_N \) \( \mathbb{P} \)-almost surely where \( \tau_N = \inf\{t \geq 0 : \|X(t)\|_{H^r} \geq N\} \). Now we aim to attain an \( H^s(\mathbb{R}^3) \)-estimate very similarly to the Proposition 4.3, we consider \( J^s \), apply \( J^s \) and then Ito’s formula with \( f(t, x) = x^2 \),
integrate over $\mathbb{R}^3$ to obtain

$$\|X(t)\|_{H^s}^2 = \|X(0)\|_{H^s}^2 + 2 \int_0^t (J^s X(\theta-), [-AJ^s X + J^s[-B(X(\theta)) - HX(\theta)])d\theta$$

$$+ 2 \int_0^t (J^s X(\theta-), J^s \Phi(X(\theta))dW(\theta)) + \int_0^t \|J^s \Phi(X(\theta))\|_{L^2}^2 d\theta$$

$$+ \int_0^t \int_Z (2 J^s X(\theta-), J^s \gamma(X(\theta-), z)),N(d\theta, dz)$$

$$+ \int_0^t \int_Z \|J^s \gamma(X(\theta-), z)\|_{L^2}^2, N(d\theta, dz).$$

Within the first integral, we compute

$$(2 J^s X, -AJ^s X + J^s[-B(X) - HX])$$

$$\leq -2\eta\|\nabla b\|_{H^s}^2 + c(\|\nabla u\|_{L^\infty} \|J^s u\|_{L^2}^2$$

$$+ (\|\nabla u\|_{L^\infty} \|J^s b\|_{L^2} + \|J^s u\|_{L^2} \|\nabla b\|_{L^\infty}) \|J^s b\|_{L^2}$$

$$+ J^s b\|_{L^2} \|\nabla b\|_{L^\infty} \|J^s u\|_{L^2}$$

$$- (\|\nabla b\|_{L^\infty} \|J^s u\|_{L^2} + \|J^s b\|_{L^\infty} \|\nabla b\|_{L^2} \|J^s u\|_{L^2})$$

$$\leq -\eta\|\nabla b\|_{H^s}^2 + c(\|\nabla X\|_{L^\infty} + \|\nabla X\|_{L^2}^2) \|X\|_{H^s}^2$$

very similarly to our estimates that led to (56) using Lemma 3.1 and Young’s inequality so that applying this estimate, taking absolute values, supremum over $t \in [0, \delta \wedge \tau_\eta]$ on the right and then left side, and expectation $E$ lead to

$$E[\sup_{t \in [0, \delta \wedge \tau_\eta]} \|X(t)\|_{H^s}^2] + \eta E[\int_0^{\delta \wedge \tau_\eta} \|\nabla b\|_{H^s}^2 d\theta]$$

$$\leq E[\|X_0\|_{H^s}^2] + cE[\int_0^{\delta \wedge \tau_\eta} (\|\nabla X\|_{L^\infty} + \|\nabla X\|_{L^2}^2) \|X\|_{H^s}^2 d\theta]$$

$$+ 2E[\sup_{t \in [0, \delta \wedge \tau_\eta]} \left| \int_0^t (J^s X(\theta-), J^s \Phi(X(\theta))dW(\theta)) \right|]$$

$$+ E[\int_0^{\delta \wedge \tau_\eta} \|\Phi(X(\theta))\|_{L^2(\mathcal{H}, H^s)}^2 d\theta]$$

$$+ E[\sup_{t \in [0, \delta \wedge \tau_\eta]} \left| \int_0^t \int_Z (2 J^s X(\theta-), J^s \gamma(X(\theta-), z)),N(d\theta, dz) \right|]$$

$$+ E[\sup_{t \in [0, \delta \wedge \tau_\eta]} \left| \int_0^t \int_Z \|\gamma(X(\theta-), z)\|_{H^s}^2, \lambda(dz) d\theta \right|]$$

$$\triangleq E[\|X_0\|_{H^s}^2] + XXII + XXIII + XXIV + XXV + XXVI.$$

We may estimate within $XXII$,

$$\|\nabla X\|_{L^\infty} + \|\nabla X\|_{L^2}^2 \lesssim (1 + \|X\|_{H^s}^2) \lesssim 1 + N^2$$

for all $t \leq \tau_\eta$ by Young’s inequality, and the fact that $s > \frac{7}{2}$. We may also estimate

$$XXIV + XXVI \lesssim E[\int_0^{\delta \wedge \tau_\eta} 1 + \|X\|_{H^s}^2 d\theta]$$

(104)
by \cite{11}. We also estimate

\[ XXIII + X X V \]

\[
\leq \mathbb{E}\left( \int_0^{\delta \land T_N} \left| \langle J^* X(\theta^-), J^* \Phi(X(\theta)) \rangle \right|^2 d\theta \right)^{\frac{1}{2}}
\]

\[ + \mathbb{E}\left( \int_0^{\delta \land T_N} \int_Z \left| \langle J^* X(\theta^-), J^* \gamma(X(\theta^-), z) \rangle \right|^2 \lambda(dz) d\theta \right)^{\frac{1}{2}} \]

\[ \leq \mathbb{E}\left( \int_0^{\delta \land T_N} \| X(\theta^-) \|_H^2 (1 + \| X(\theta) \|_H^2) d\theta \right)^{\frac{1}{2}} \]

by Burkholder-Davis-Gundy inequality, Hölder’s inequality, \cite{11} and Young’s inequality. After applying \cite{103} - \cite{105} to \cite{102} and subtracting \( \frac{1}{2} \mathbb{E}[\sup_{t \in [0, \delta \land T_N]} \| X(t) \|_H^2] \), Gronwall’s inequality gives for some constant \( c_1 > 0 \),

\[ \mathbb{E}\left[ \sup_{t \in [0, \delta \land T_N]} \| X(t) \|_H^2 \right] + c_2 \mathbb{E}\left[ \int_0^{\delta \land T_N} \| \nabla b \|_H^2 d\theta \right] \]

\[ \leq (1 + 2 \mathbb{E}[\| X_0 \|_H^2]) e^{c_1(1 + N^2) \delta}. \]

Now for the fixed \( \delta \in (0, 1) \), we find \( N \in \mathbb{N} \) such that

\[ \frac{1}{N + 1} \leq \delta^\frac{1}{2} < \frac{1}{N}. \]

It is clear that

\[ \{ \omega \in \Omega : \sup_{t \in [0, \delta \land T_N]} \| X(t) \|_H^2 < N \} \subset \{ \omega \in \Omega : \tau_N(\omega) > \delta \} \]

while

\[ \{ \omega \in \Omega : \tau_N > \delta \} \subset \{ \omega \in \Omega : \tau > \delta \} \]

because \( \lim_{M \to \infty} \tau_M = \tau \). Therefore, \cite{13} may be proven as follows to complete the proof of Theorem 2.2

\[ \mathbb{P}(\{ \tau > \delta \}) \geq \mathbb{P}(\{ \sup_{t \in [0, \delta]} \| X(t \land \tau_N) \|_H^2 < N \}) \]

\[ \geq 1 - \frac{\mathbb{E}[\sup_{t \in [0, \delta]} \| X(t \land \tau_N) \|_H^2]}{N^2} \]

\[ \geq 1 - \frac{(1 + 2 \mathbb{E}[\| X_0 \|_H^2]) e^{c_1(1 + N^2) \delta}}{N^2} \geq 1 - c_0 \delta (1 + 2 \mathbb{E}[\| X_0 \|_H^2]) \]

for \( c_0 > 0 \) independent of \( \delta, X_0 \) if we take \( c_0 > 4 e^{2c_1} \) where we used \cite{109}, \cite{108}, Chebyshev’s inequality, \cite{106} and that \( \frac{e^{c_1(1 + N^2) \delta}}{N^2} \leq c_0 \delta \) which follows from \cite{107}.

6. PROOF OF THEOREM 2.3

We fix \( \epsilon > 0 \) arbitrary small and define

\[ \tau_\delta(\omega) \triangleq \inf\{ t \geq 0 : \| X(t) \|_H^2 \geq \delta \} \]

(110)
for $\delta \in (0, 1)$, and $Y(t) \triangleq \|X(t \wedge \tau_0)\|_H^2$. Let us denote by $c_0 > 0$ the general constant such that the inequality (17) holds. Then we may denote by $c > 0$ the general constant such that

$$(30c_0\|\nabla f\|_{L^\infty} + 4c_0^2\|\nabla f\|_{L^2}^2) \leq c(\|f\|_{H^{1.5}} + \|f\|_{H^{3/2}}^2)$$

so that such $c > 0$ is independent of $f$. Now by hypothesis, $\lambda(z) < \infty$ and $(\ln(2))^{\lambda(z)} < K_2^2$ due to (14a). It may be shown that for $\alpha > 0$ sufficiently small,

$$(2^\alpha - 1)\lambda(Z) < 2\alpha(1 - \alpha)^2K_1^2$$

and therefore, for the fixed constant $c > 0$, we may fix $\alpha = \alpha(K_1, K_2, \lambda(Z)) > 0$ such that

$$\alpha(c \alpha \delta + \alpha K_2^2) + (2^\alpha - 1)\lambda(Z) < 2\alpha(1 - \alpha)^2K_1^2.$$ 

With such $c > 0$ and $\alpha = \alpha(K_1, K_2, \lambda(Z))$ fixed, let us find $K_3 = K_3(\alpha, \lambda(Z)) = K_3(K_1, K_2, \lambda(Z)) > 0$ such that

$$\alpha(c \alpha \delta + \alpha K_2^2) - 2\alpha(1 - \alpha)^2K_1^2 + [(2^\alpha - 1) + 2\alpha(2K_3 + K_3^2)]^\alpha + 2\alpha K_3\lambda(Z) \leq 0.$$

Now let us assume $\mathbb{E}[\|X_0\|_{H^{1.5}}^4] \leq \mu \triangleq \epsilon^2 \delta^4$ and denote

$$\theta(t, X, z) \triangleq \chi_{[0, \tau_0]}(2(J^\alpha X, J^\alpha \gamma(X(t-), z)) + \|\gamma(X(t-), z)\|_{L^2}^2).$$

(111)

We apply $J^\alpha$ on the differential form of (10), and Ito’s formula with $f(t, x) = x^2$, integrate over $\mathbb{R}^3$, apply Ito’s formula on the resulting equation again with $f(t, x) = (\zeta + x)^\alpha$, take expectation $\mathbb{E}$ to deduce

$$\mathbb{E}[(\zeta + Y(t))^{\alpha}] - \mathbb{E}[(\zeta + Y(0))^{\alpha}]$$

$$= \alpha \mathbb{E}\left[\int_0^t (\zeta + Y(-))^{\alpha-1}\chi_{[0, \tau_0]}[(2J^\alpha X, [-AJ^\alpha X - J^\alpha B(X) - J^\alpha HX])] + \|\Phi(X(\xi \wedge \tau_0))\|_{L^2(\mu, H^{1.5})} + \int_{\mathbb{R}^3} \|\gamma(X(\xi \wedge \tau_0), z)\|_{L^2}^2 \lambda(\{dz\}) d\xi\right]$$

$$+ 2\alpha(1 - \alpha) \mathbb{E}\left[\int_0^t (\zeta + Y(\xi))^{\alpha-2}\chi_{[0, \tau_0]}((J^\alpha X(\xi), J^\alpha \Phi(X(\xi))))^2 d\xi\right]$$

(112)

$$+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^3} (\zeta + Y(\xi - \theta(\xi, X, z)))^{\alpha - (\zeta + Y(\xi - \theta(\xi, X, z))^{\alpha - 1}\lambda(\{dz\}) d\xi].$$

We estimate similarly to the computations that led (56) to (57) (see also (101))

$$(2\alpha J^\alpha X, [-AJ^\alpha X - J^\alpha B(X) - J^\alpha HX])$$

$$\leq - 2\alpha\eta\|J^\alpha \nabla b\|_{L^2}^2$$

$$+ 4c_0\alpha(\|\nabla u\|_{L^\infty} + \|J^{1.5} u\|_{L^2})^2 + \|\nabla u\|_{L^\infty} + \|J^\alpha b\|_{L^2}^2 + \|J^\alpha u\|_{L^2} + \|\nabla b\|_{L^\infty} + \|J^{1.5} b\|_{L^2}) (113)$$

$$+ c_0(\|J^{1.5} b\|_{L^2} + \|J^\alpha b\|_{L^2})\|J^\alpha \nabla b\|_{L^2}$$

$$\leq - \alpha\eta\|J^\alpha \nabla b\|_{L^2}^2 + \frac{c_0}{2}(\|X\|_{H^{1.5}} + \|X\|_{H^{3/2}}^2) \|X\|_{L^2} \leq - \alpha\eta\|J^\alpha \nabla b\|_{L^2}^2 + c\alpha \delta Y$$
Applying (113) to (112), and using the hypothesis of (14a) and (14b) give

t for all

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due to [27, (7.4)]. Using (116) we may compute

We may estimate

Therefore, we finally deduce using our choice of

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Therefore, we finally deduce using our choice of \( c_0 > 0, c > 0 \) and (110).

Applying (113) to (112), and using the hypothesis of (14a) and (14b) give

\[
\begin{align*}
\mathbb{E}[(\zeta + Y(t))^\alpha] + \alpha \eta \mathbb{E} \left[ \int_0^{t \wedge \tau_3} (\zeta + Y(\xi -))^\alpha \| J^* \nabla b \| L_2^2 d\xi \right] \\
\leq \mathbb{E}[(\zeta + Y(0))^\alpha] + \mathbb{E} \left[ \int_0^{t \wedge \tau_3} (\zeta + Y(\xi -))^\alpha \| (c\alpha + \alpha K_2^2) Y \| d\xi \right] \\
- 2\alpha(1 - \alpha)K_1^2 \mathbb{E} \left[ \int_0^{t \wedge \tau_3} (\zeta + Y(\xi -))^{\alpha - 2} \| Y^2 \| d\xi \right] \\
+ \mathbb{E} \left[ \int_0^{t \wedge \tau_3} (\zeta + Y(\xi -))^{\alpha} - (\zeta + Y(\xi -))^\alpha \\
- 2\alpha(J^* X, J^* \gamma(X(\xi -), z))(\zeta + Y(\xi -))^\alpha \lambda(dz) d\xi \right].
\end{align*}
\]

We may estimate

\[
(\zeta + Y(\xi -) + \theta(\xi, X, z))^\alpha - (\zeta + Y(\xi -))^\alpha \\
- 2\alpha(J^* X, J^* \gamma(X(\xi -), z))(\zeta + Y(\xi -))^\alpha - 1
\]

\[
\leq (2^\alpha - 1)|\zeta + Y(\xi -)|^\alpha \\
+ 2^\alpha |2\gamma(X(\xi -), z)||H^*|X||H^* + ||\gamma(X(\xi -), z)||_H^2|d\xi |
\]

\[
2\alpha(\zeta + Y(\xi -))^{\alpha - 1}||\gamma(X(\xi -), z)||_H^2||X||_H^2 \\
\leq (2^\alpha - 1)|\zeta + Y(\xi -)|^\alpha + 2^\alpha (2K_3\|X\|_H^2 + K_3^2\|X\|_H^2)^\alpha \\
+ 2\alpha(\zeta + Y(\xi -))^{\alpha - 1}K_3\|X\|_H^2 \\
\leq (2^\alpha - 1) + 2^\alpha (2K_3 + K_3^2)^\alpha + 2\alpha K_3(\zeta + Y(\xi -))^\alpha.
\]

Now we may use

\[
b^2 \geq (1 - \alpha)(b + c)^2 - \left( \frac{1 - \alpha}{\alpha} \right) c^2 \quad \forall \ b, c \geq 0, \alpha \in (0, 1)
\]

due to [27, (7.4)]. Using (116) we may compute

\[-2\alpha(1 - \alpha)K_1^2(\zeta + Y)^{\alpha - 2}Y^2 \leq -2\alpha(1 - \alpha)^2K_1^2(\zeta + Y)^\alpha + 2\alpha \left( \frac{1 - \alpha}{\alpha} \right) K_1^2 \zeta^\alpha.
\]

Using this and (115) in (114) leads to

\[
\mathbb{E}[(\zeta + Y(t))^\alpha] + \alpha \eta \mathbb{E} \left[ \int_0^{t \wedge \tau_3} (\zeta + Y(\xi -))^\alpha \| J^* \nabla b \| L_2^2 d\xi \right] \\
\leq \mathbb{E}[(\zeta + Y(0))^\alpha] + 2(1 - \alpha)^2K_1^2\zeta^\alpha t \\
+ |\alpha(c\alpha + \alpha K_2^2) - 2\alpha(1 - \alpha)^2K_1^2 + [(2^\alpha - 1) + 2^\alpha (2K_3 + K_3^2)^\alpha + 2\alpha K_3]\lambda(z) |
\]

\[
\times \mathbb{E} \left[ \int_0^{t \wedge \tau_3} (\zeta + Y(\xi -))^\alpha d\xi \right].
\]

Therefore, we finally deduce using our choice of \( c_0, c, \alpha \) and \( K_3 \), after passing \( \zeta \to 0^+ \)

\[
\mathbb{E}[\| X(t \wedge \tau_3) \|^2_{H^\alpha}] \leq \mathbb{E}[\| X_0 \|^2_{H^\alpha}],
\]

(117)
Thus, for a set \( \Theta \triangleq \{ \omega \in \Omega : \tau_\delta(\omega) < \infty \} \), we deduce

\[
\mathbb{P}(\Theta) \leq \frac{1}{\delta^{2\alpha}} \mathbb{E}[\liminf_{t \to \infty} \chi_\Theta \|X(t \wedge \tau_\delta)\|^{2\alpha}_{H^s}]
\]

\[
\leq \frac{1}{\delta^{2\alpha}} \liminf_{t \to \infty} \mathbb{E}[\chi_\Theta \|X(t \wedge \tau_\delta)\|^{2\alpha}_{H^s}]
\]

\[
\leq \frac{1}{\delta^{2\alpha}} \liminf_{t \to \infty} \mathbb{E}[\|X_0\|_{H^s}^{2\alpha}] \leq \frac{1}{\delta^{2\alpha}} \left( \mathbb{E}[\|X_0\|_{H^s}^{4\alpha}] \right) \frac{\delta}{2} \leq \epsilon
\]

by our choice \( \mathbb{E}[\|X_0\|_{H^s}^{4\alpha}] \leq \epsilon^2 \delta^4 \), Chebyshev’s inequality, Fatou’s theorem, (117), and Hölder’s inequality. This leads to (15) and thus the proof of Theorem 2.3 is complete.

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