Abstract: Micropolar fluids have microstructure and belong to a class of fluids with nonsymmetric stress tensor. We study the incompressible two-dimensional micropolar fluid system with periodic boundary condition forced by random noise that is white-in-time. In particular, we obtain a sufficient condition on the size of the angular viscosity coefficients in comparison to the vortex and kinematic viscosity coefficients so that the solution to this system is smooth in the Malliavin sense. In addition, we prove a result concerning an orthogonal projection onto a finite number of Fourier modes, taking advantage of the dissipative nature of the system.

Keywords: Dissipativity, Malliavin derivatives, micropolar fluid, Navier–Stokes equations, strong Feller

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1 Introduction

The Navier–Stokes equation (NSE) has a limitation in that it cannot model fluids with microstructure, although complex fluids such as polymeric suspensions, animal blood or liquid crystals possess the need to take into account such an aspect because individual particles of such fluids may have different shape, shrink, expand or even rotate independently of the rotation and movement of the fluid. More than half a century ago, the theories of simple microfluids and micropolar fluids (MPF) were introduced by Eringen in [12, 13], respectively. In particular, the micropolar continua, together with those of micromorphic and microstretch, constitute the microcontinuum field theories (see [14, 15, 25]). Due to such a generality of the MPF compared to the NSE, many scientists not only in mathematics (e.g., [3, 7–9, 21, 38, 40, 43]) but physics and engineering have devoted much effort to its research.

The ergodicity of the NSE and related equations, specifically the existence of a unique invariant measure which is ergodic, has received much attention from many mathematicians for decades. In particular, the works [6, 17, 19] proved the existence of an invariant measure for the Burgers equation, Bénard problem and NSE, respectively, and the works [16, 18, 20] were devoted to the uniqueness of the invariant measure for the NSE (see also [41] for the work on the magnetic Bénard problem). Within the proof of ergodicity of an infinite-dimensional system, the regularity issue of the transition density of the solution process is crucial in verifying the strong Feller property of the transition semigroup generated by the solution process. On the other hand, stochastic calculus of variations, also known as Malliavin calculus, is an infinite-dimensional differential calculus on the Wiener space (see [2, 30]) initiated by Malliavin in [26, 27], which introduced the Malliavin covariance matrix (see [27, (5.3.2)]). Analysis on this matrix may be used to deduce existence...
as well as regularity of the density (see, e.g., [35, Corollary (1.21)] and [34, Theorem 3.9]); in short, if $F$ is a random vector such that each component is smooth in the Malliavin sense, $y_F$ is the Malliavin covariance matrix of $F$ and $|\det y_F|^{-1}$ has finite expectation for all $p > 1$, then $F$ has an infinitely differentiable density.

The purpose of this manuscript is to execute Malliavin calculus on the MPF system and in particular obtain sufficient conditions on the size of viscosity coefficients so that the solution is smooth in the Malliavin sense. Applications of Malliavin calculus in the finite-dimensional case may be found in [10, 24, 32]. In contrast, the MPF system of our concern is infinite-dimensional (see (2.1a), (2.1b)), and the coupling within the system makes it considerably more difficult than the case of the NSE. Moreover, we will also point out a difficulty in extending the ergodicity results of [22, 29] to the MPF system. In fact, our explanation applies to almost any other system of equations that is coupled pointing out the difficulty in extending the proofs of [22, 29] in a straightforward way (see Remark 2.4). With this in mind, we choose to follow the notations of [22, 29], although some generalizations may be possible. Finally, we note that ergodicity results for the MPF system were obtained in [42]; however, in contrast to [22, 29], all the Fourier modes of the noise term in [42] had to be forced.

## 2 Statement of main results

We consider the spatial domain to be a two-dimensional (2D) torus $\mathbb{T}^2 = [0, 2\pi]^2$. We denote by $(u_1, u_2, u_3)$, $(w_1, w_2, w_3)$ and $p$ the velocity, micro-rotational velocity and pressure fields, respectively, and represent by $\chi$, $\mu$, $\nu > 0$, the vortex, kinematic and angular viscosities, respectively. We follow the appropriate adjustment in the 2D case (see [25, p. 185]) and set $u \equiv (u_1, u_2, 0)$ and $w \equiv (0, 0, w_3)$, due to which $\nabla \cdot w = 0$ on $\mathbb{T}^2$. Finally, we let $\xi_u$, $\xi_w$ be random forcing terms to be elaborated subsequently, with which we may introduce the micropolar fluid (MPF) system

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = (\mu + \chi) \Delta u + \chi \nabla \times w + \xi_u, \tag{2.1a}
\]

\[
\frac{\partial w}{\partial t} + (u \cdot \nabla) w = \nu \Delta w + \chi \nabla \times u + \xi_w. \tag{2.1b}
\]

Let us immediately note that at $w = 0$ system (2.1a)–(2.1b) reduces to the NSE. For simplicity, we write $\frac{\partial}{\partial t}, \int f$ for $\frac{\partial}{\partial t}, \int f(x) \, dx$, and $\nu \equiv \nabla \times u$, $\nu \nu = u$, i.e. $\nu$ is the Biot–Savart law. We consider a Hilbert space of

\[
\begin{align*}
\mathcal{E}_0^2 & \triangleq \left\{ f : \int f = 0, \int |f|^2 < \infty \right\},
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{Z}_1^2 & \triangleq \{ k = (k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0 \} \cup \{ k = (k_1, k_2) \in \mathbb{Z}^2 : k_1 > 0, k_2 = 0 \},
\end{align*}
\]

\[
\begin{align*}
\mathcal{Z}_2^2 & \triangleq -\mathcal{Z}_1^2, \quad \text{so that } \mathcal{Z}_0^2 \triangleq \mathcal{Z}_1^2 \cup \mathcal{Z}_2^2 \setminus \{(0, 0)\}. \quad \text{We define the convenient basis}
\end{align*}
\]

\[
\begin{align*}
E_k(x) & \triangleq \hat{E}_k(x) \triangleq \begin{cases} 
\sin(k \cdot x) & \text{if } k \in \mathcal{Z}_1^2, \\
\cos(k \cdot x) & \text{if } k \in \mathcal{Z}_2^2,
\end{cases}
\end{align*}
\]

and represent the forcing terms in the forms of

\[
\begin{align*}
N_u(x, t) & \triangleq \sum_{k \in \mathcal{Z}_1^2} \sigma_k^u n_k^u(t) E_k(x), \quad N_w(x, t) \triangleq \sum_{k \in \mathcal{Z}_2^2} \sigma_k^w n_k^w(t) \hat{E}_k(x),
\end{align*}
\]

where $\mathcal{Z}_1^w, \mathcal{Z}_2^w \subset \mathcal{Z}_0^2$, $\{n_k^u\}_{k \in \mathcal{Z}_1^2}$, $\{n_k^w\}_{k \in \mathcal{Z}_2^2}$ are mutually independent standard Wiener processes on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and $\{|\sigma_k^u|\}_{k \in \mathcal{Z}_1^2}$, $\{|\sigma_k^w|\}_{k \in \mathcal{Z}_2^2}$ represent noise intensity. We furthermore denote by $E$ the expectation with respect to $\mathbb{P}$.

We now consider the following system of equations equivalent to (2.1a)–(2.1b):

\[
\begin{align*}
\frac{dv}{dt} & = -((\chi \nu) \cdot \nabla)v + (\mu + \chi) \Delta v - \chi \Delta w \, dt + dN_v, \quad v(0) = v(0), \tag{2.3a}
\end{align*}
\]

\[
\begin{align*}
\frac{dw}{dt} & = -((\chi \nu) \cdot \nabla)w - 2\chi w + \chi w \nabla \Delta w + \chi v \, dt + dN_w, \quad w(0) = w(0). \tag{2.3b}
\end{align*}
\]
where we write
\[ v = \sum_{k} a_k(t)e_k(x), \quad w = \sum_{k} \beta_k(t)\hat{e}_k(x), \quad \mathcal{K}v(x, t) = \sum_{k} \frac{k^2}{|k|^2} a_k(t)e_k(x), \] (2.4)
and also \( B(f, g) \equiv (f \cdot V)g \). We let \( \| \cdot \| \) denote the norm of \( \mathbb{L}_0^2 \), while \( \| \cdot \|_{2} \) is the norm of \( \mathbb{L}^2 \times \mathbb{L}^2 \) and \( \langle \cdot , \cdot \rangle \) its inner product. We set \( \mathbb{H}^s \), \( s \in \mathbb{R} \) with its norm \( \| \cdot \|_{\mathbb{H}^s} = \| \mathcal{F} \cdot \|_{\mathbb{L}^2} \), where \( \mathcal{F} = (-\Delta)^{s/2} \) is defined via the Fourier transform as \( \mathcal{F} f(k) = |k|^s f(k) \); we recall that by Poincaré’s inequality, such a homogeneous Sobolev norm is equivalent to that of the inhomogeneous Sobolev norm of \( \mathbb{H}^s \).

In preparation to state our results, let us review basic definitions of Malliavin calculus and set notations. We let \( \mathcal{H} \) be any separable Hilbert space, \( \Theta = C([0, t]; \mathbb{R}^d), d \in \mathbb{N} \), and suppose that there exists a continuous Markovian stochastic semiflow on \( \mathcal{H} \) generated by a stochastic system given by \( \Psi_t : \Theta \times \mathcal{H} \mapsto \mathcal{H} \), specifically \( \Psi_t([N_i]_{i=1}^d, f) \), where \( [N_i]_{i=1}^d \in \Theta \) is a set of mutually independent Brownian motions on \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( f \in \mathcal{H} \). We let
\[ H \equiv \left\{ x \in \Theta : x_i(t) = \int_0^t \partial_s x_i ds \text{ for } i = 1, \ldots, d, \int_0^t \partial_s x_i^2 ds < \infty \right\}, \]
equipped with an inner product of
\[ \langle x, y \rangle_H \equiv \int_0^t \partial_s x_i \partial_s y_i ds; \]
such a space \( H \) is called a Cameron–Martin space (see [30, p. 25]). We denote by \( HS(X; Y) \) the space of Hilbert–Schmidt operators from a Hilbert space \( X \) to another Hilbert space \( Y \). For \( F : \Theta \mapsto \mathcal{H} \), let us write \( F \in D^1(\Theta) \) and say that \( F \) has a Malliavin derivative \( D^1 F : \Theta \mapsto HS(H; \mathcal{H}) \); if for all \( y(t) = \int_0^t \lambda(s) ds \in H \) such that \( \int_0^t |\lambda(s)|^2 ds < \infty \) there is a version \( F_y \) of \( F \) such that the mapping \( e \mapsto F_y(\theta + ey) \) is absolutely continuous for all \( \theta \in \Theta \) and
\[ \mathbb{E}\left[ \left| F(\theta + ey) - F(\theta) \right| e \mathbb{D} F(\theta) \right] \to 0 \quad \text{as } e \to 0 \quad \text{for all } y \in H. \]

For \( F, G \in D^1(\Theta) \), we define their Malliavin covariance matrix
\[ \nabla F^* \nabla G(\theta) \equiv \sum_{n=1}^{\infty} D^1 F(\theta) \otimes D^1 G(\theta), \]
where \( \{h_n\}_{n=1}^{\infty} \) is an orthonormal basis of \( H \) and \( h_n(t) = \int_0^t \eta_i(s) ds \).

Now it can be shown that system (2.3a)–(2.3b) generates a continuous Markovian semiflow on \( \mathbb{L}^2 \),
\[ \Phi_t : C([0, t]; \mathbb{R}^d) \times C([0, t]; \mathbb{R}^d) \mapsto \mathbb{L}^2, \]
defined by
\[ \Phi_t((n_k^v)_{k \in \mathbb{Z}^*}, (n_k^w)_{k \in \mathbb{Z}^*}, (v_0, w_0)) = (\Phi^v_t, \Phi^w_t)((n_k^v)_{k \in \mathbb{Z}^*}, (n_k^w)_{k \in \mathbb{Z}^*}, (v_0, w_0)) = (v, w)(t), \]
that solves the stochastic MPF system at time \( t \) starting from \((v_0, w_0)\) at time \( t = 0 \) (see [38, 40, 42]). We denote
\[ \text{card}(Z^v) \equiv |Z^v|, \quad \text{card}(Z^w) \equiv |Z^w|, \quad N \equiv |Z^v| + |Z^w|, \quad Z^v \equiv [1, \ldots, |Z^v|], \quad Z^w \equiv |Z^v| + 1, \ldots, N, \]
and fix \( \{n_1, \ldots, n_{|Z^v|}\} \) and \( \{n_{|Z^v|+1}, \ldots, n_N\} \) as any possible ordered set of \( \{n_k^v\}_{k \in Z^v} \) and \( \{n_k^w\}_{k \in Z^v} \), respectively. We furthermore define an operator \( Q : \mathbb{R}^N \mapsto \mathbb{L}^2 \) by
\[ Q(x_1, \ldots, x_N) \equiv \left( \sum_{i \in Z^v} x_i e_i, \sum_{i \in Z^w} x_i \tilde{e}_i \right), \]
where \( n_i e_i = n_k^v e_k \) if \( i \in Z^v \) and \( n_i \tilde{e}_i = n_k^w \tilde{e}_k \) if \( i \in Z^w \), for \( k \) corresponding to \( n_i \) so that we may write
\[ (\Phi^v_t, \Phi^w_t)((n_i^v)_{i=1}^N, (v_0, w_0)) = (v, w)(t). \]
We denote the standard basis of \( \mathbb{R}^N \) by \( \{ q_i \}_{i=1}^N \) and define \( D^{i,\lambda}(v, w) \), where \( \int_0^t \lambda(s) \, ds = \gamma(t) \) and
\[
E \left[ \frac{\Phi_i((n_1, \ldots, n_N), q_1) - \Phi_i((n_1, \ldots, n_N))}{\varepsilon} - D^{i,\lambda}(v, w) \right] \to 0 \quad \text{as } \varepsilon \to 0.
\]

Consider
\[
(V_{k,s}^w, V_{k,s}^w) \in C([s, +\infty); \mathbb{L}^2) \cap L^2_{ loc}([s, +\infty); \mathbb{H}^1)
\]
that solves
\[
\begin{align*}
\partial_t V_{k,s}^w(t) &= (\mu + \chi) \Delta V_{k,s}^w - ((\mathcal{X}v) \cdot \nabla) V_{k,s}^w - ((\mathcal{X} V_{k,s}^w) \cdot \nabla) v - \chi \Delta V_{k,s}^w, \quad s < t, \\
\partial_t V_{k,s}^w(t) &= -2 \chi V_{k,s}^w + \chi \Delta V_{k,s}^w - ((\mathcal{X}v) \cdot \nabla) V_{k,s}^w - ((\mathcal{X} V_{k,s}^w) \cdot \nabla) w + \chi \Delta V_{k,s}^w, \quad s < t,
\end{align*}
\]
\[
V_{k,s}(s) = (V_{k,s}^w, V_{k,s}^w)(s) = (e_k, \bar{e}_k),
\]
where for all \( s \in [0, t] \), \( k \in \mathbb{Z}^+_v \cup \mathbb{Z}^+_w \) and for all \( h \in L^2_{ loc}(\mathbb{R}_+) \) we have
\[
D^{k,h} v(t) \equiv \int_0^t V_{k,s}^w(h(s)) \, ds, \quad D^{k,h} w(t) \equiv \int_0^t V_{k,s}^w(h(s)) \, ds,
\]
and hence
\[
D^{k,v}(t) \equiv D^{k,\delta v}(t) = V_{k,s}^w(t), \quad D^{k,w}(t) \equiv D^{k,\delta w}(t) = V_{k,s}^w(t).
\]
We define \( J^{v}_{s,t}(\phi^v, \phi^w), \) where \( J^{v}_{s,t}(Q_{k,s}) = V_{k,s}^w \) and \( J^{w}_{s,t}(Q_{k,s}) = V_{k,s}^w \), and consider the following system of equations with \( \phi = (\phi^v, \phi^w) \in \mathbb{L}^2 \):
\[
\begin{align*}
\partial_t J^{v}_{s,t}(\phi) &= (\mu + \chi) \Delta J^{v}_{s,t}(\phi) - ((\mathcal{X}v) \cdot \nabla) J^{v}_{s,t}(\phi) - ((\mathcal{X} J^{v}_{s,t}(\phi)) \cdot \nabla) v - \chi \Delta J^{v}_{s,t}(\phi), \quad 0 \leq s \leq t, \\
\partial_t J^{w}_{s,t}(\phi) &= -2 \chi J^{w}_{s,t}(\phi) + \chi \Delta J^{w}_{s,t}(\phi) - ((\mathcal{X}v) \cdot \nabla) J^{w}_{s,t}(\phi) - ((\mathcal{X} J^{w}_{s,t}(\phi)) \cdot \nabla) w + \chi \Delta J^{w}_{s,t}(\phi), \quad 0 \leq s \leq t,
\end{align*}
\]
\[
J^{v}_{s,s}(\phi) = (\phi^v, \phi^w) = \phi.
\]
We finally state our main result.

**Theorem 2.1.** (i) (cf. [29, Lemma B.1]) Suppose
\[
\max \left[ \frac{\chi^2}{2(\mu + \frac{1}{2})}, \frac{\chi}{2} \right] < \gamma.
\]
Then for any \( T_0 > 0, \alpha = 0 \) or 1, \( \eta > 0 \), there exist constants \( \theta > 0 \) sufficiently small such that for any \( \phi = (\phi^v, \phi^w) \in \mathbb{L}^2, T \leq T_0 \), the solution \( (J^{v}_{s,t}(\phi), J^{w}_{s,t}(\phi)) \) to (2.5a)–(2.5c) satisfies
\[
\sup_{0 \leq s \leq T} (\| J^{v}_{s,t}(\phi) \|_{L^2}^2 + \| J^{w}_{s,t}(\phi) \|_{L^2}^2 + \Theta(t-s) \| J^{v}_{s,t}(\phi) \|_{H^1}^2 + \| J^{w}_{s,t}(\phi) \|_{H^1}^2)) \leq e^{\eta \int_0^T \| v \|_{L^2}^2 + \| w \|_{L^2}^2} \| \phi \|_{L^2}^2 (\| \phi \|_{H^2}^2 + (1-\lambda) \| \phi \|_{H^1}^2),
\]
where \( C_1 = C_1(\Theta, \chi, \eta, T_0) > 0 \) and \( (1-\lambda) \| \phi \|_{H^1}^2 = 0 \) if \( \lambda = 1 \) for all \( \phi \) including those such that \( \| \phi \|_{H^1} = +\infty \).

(ii) Suppose
\[
\max \left[ \frac{\chi^2}{2(\mu + \frac{1}{2})}, \frac{\chi}{2} \right] < \gamma.
\]
Then for any \( T_0 > 0, \alpha = 0 \) or 1, there exist constants \( \eta > 0 \) sufficiently small, \( \theta > 0 \) sufficiently small such that for any \( \phi = (\phi^v, \phi^w) \in \mathbb{L}^2, T \leq T_0 \), the solution \( (J^{v}_{s,t}(\phi), J^{w}_{s,t}(\phi)) \) to (2.5a)–(2.5c) satisfies
\[
\sup_{0 \leq s \leq T} (\| J^{v}_{s,t}(\phi) \|_{L^2}^2 + \| J^{w}_{s,t}(\phi) \|_{L^2}^2 + \Theta(t-s) \| J^{v}_{s,t}(\phi) \|_{H^1}^2 + \| J^{w}_{s,t}(\phi) \|_{H^1}^2)) \leq C_2 e^{\Theta(\varepsilon \| v \|_{H^1}^2 + \| w \|_{H^1}^2) + C_1^T (\| \phi \|_{L^2}^2 + (1-\lambda) \| \phi \|_{H^1}^2)},
\]
where \( C_1 = C_1(\Theta, \chi, \eta, T_0) > 0, C_2 = C_2(T_0, \eta, \mu, \chi, y, \varepsilon_0, \varepsilon_0^w, \Theta = \Theta(\mu, \chi, y) \) and \( (1-\lambda) \| \phi \|_{H^1}^2 = 0 \) if \( \lambda = 1 \) for all \( \phi \) including those such that \( \| \phi \|_{H^1} = +\infty \).
Let us recall that analogously to the classical Sobolev spaces, one may define $\mathcal{D}^{k,p}$, $p \geq 1$, $k \in \mathbb{N}$, with a norm of

$$\|F\|_{k,p} = \left( E[|F|^p] + \sum_{j=1}^{k} E[\|D^j F\|_{L^p((0,T))}] \right)^{\frac{1}{p}}$$

(see [30, p. 27] for details) and

$$\mathcal{D}^{\infty} \equiv \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathcal{D}^{k,p}$$

(see [30, p. 62]). Now we state two corollaries of Theorem 2.1.

**Corollary 2.2** (cf. [29, Lemma C.1]). Suppose

$$\max \left\{ \frac{X^2}{2(\mu + \frac{1}{2})}, \frac{X}{2} \right\} < \gamma.$$ 

Then the solution to the MPF system $(v, w)(t)$ satisfies $(v, w)(t) \in \mathcal{D}^{\infty}$ for all $t > 0$.

We may also prove the following corollary which is an extension of [22, Lemma 4.17] to the MPF system and shows the dissipative nature of the MPF system.

**Corollary 2.3** (cf. [22, Lemma 4.17]). Suppose

$$\max \left\{ \frac{X^2}{2(\mu + \frac{1}{2})}, \frac{X}{2} \right\} < \gamma.$$ 

Then for any $C_0 > 0$, $p \in [1, \infty)$, $T > 0$, $\lambda > 0$, there exists an orthogonal projection $\pi_l$ onto low Fourier modes so that denoting the norm of an operator on $L^2$ by $\| \cdot \|_{\text{ope}}$, we have

$$E[\|((Id - \pi_l) f^r_{0,T}, (Id - \pi_l) f^w_{0,T})\|_{\text{ope}}^p] \leq C_0 E[\exp(\lambda \|v(0)\|_{L^2}^2 + \|w(0)\|_{L^2}^2)],$$

$$E[\|\left( f^r_{0,T}, (Id - \pi_l) f^w_{0,T} \right)\|_{\text{ope}}^\gamma] \leq C_0 E[\exp(\lambda \|v(0)\|_{L^2}^2 + \|w(0)\|_{L^2}^2)].$$

**Remark 2.4.** (i) The assumption of

$$\max \left\{ \frac{X^2}{2(\mu + \frac{1}{2})}, \frac{X}{2} \right\} < \gamma$$

may be somewhat relaxed by optimizing through the proofs. Nevertheless, some condition of $\gamma > 0$ being large relative to other viscosity coefficients seems to be unavoidable; we chose to present our assumption for simplicity. Such a condition to guarantee the smoothness of Malliavin derivatives is very reminiscent of the typical condition to assure the global regularity of solutions to deterministic equations in fluid dynamics when the initial data is sufficiently small in comparison to the viscosity coefficient (see, e.g., [36, Theorem 4.1, p. 393]). We also recall the important work of [28] in which the author proved that a certain assumption on the size of the viscosity coefficient implies that the 2D NSE has a unique invariant measure.

(ii) One of the important motivations to study the MPF system is to investigate its similarities and differences to the Boussinesq system

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \mu \Delta u + \theta e_2,$$  \hspace{1cm} (2.8a)

$$\partial_t \theta + (u \cdot \nabla) \theta = \eta \Delta \theta,$$  \hspace{1cm} (2.8b)

where $\theta$ is a scalar temperature field and $\eta > 0$ is the thermal diffusivity. There are two additional difficulties in the case of the MPF system (2.1a)–(2.1b), in contrast to the Boussinesq system (2.8a)–(2.8b); not only is the $\chi \nabla \times w$ in (2.1a) one more derivative more singular than $\theta e_2$ in (2.8a), there is the presence of $\chi \nabla \times u$ in (2.1b). For this precise reason, although Hmidi, Keraani and Rousset [23] were able to prove the global well-posedness of the deterministic Boussinesq system with $\eta = 0$, $\mu > 0$ and $-\mu u$ replaced by $\mu (-\Delta)^{1/2}$, such a result is absent in the case of the MPF system (see [9, 39]). Our assumption on the size of the viscosity coefficient may be seen as the smallness of $\chi$ with which our estimates will be allowed to go through similarly to the case of the Boussinesq system.
In an effort to eliminate the condition of
\[
\max \left\{ \frac{X^2}{2(\mu + \frac{1}{2}Y)}, \frac{X}{2} \right\} < \gamma,
\]
we tried considering \((u, w)\) or \((v, \nabla w)\) instead of \((v, w)\) in (2.3a)–(2.3b) because it seems unbalanced due to the term \(-X\Delta w\) in (2.3a) and \(\gamma\Delta w\) in (2.3b), but faced difficulties in both cases. With this in mind, it is not clear to the author whether Theorem 2.1 and Corollaries 2.2 and 2.3 may be extended to the stochastic magnetohydrodynamics (MHD) system that has also caught much attention (e.g., [1, 4, 33]).

(iii) As mentioned in Section 1, our work was initially motivated by [22, 29], which may be considered as nonlinear versions of the work [31]. Most importantly, our work was initially inspired by the work [37]; however, we could not follow the argument therein, in particular due to the lack of symmetry in (2.1b) as well as (2.8b) in contrast to the NSE, as we elaborate more below.

(iv) Due to Theorem 2.1 and [30, Theorem 2.1.2], if we could show that there exists a fixed subset \(\Omega_1\) of positive probability such that \(\phi \in L^2\) and \((\gamma(t)\phi, \phi) = 0\) on any realization in \(\Omega_1\), where \(\gamma(t)\) is the Malliavin covariance matrix of \(F(t) = (v, w)(t)\), then this implies that for any \(t > 0\) and any finite-dimensional subspaces \(S_1, S_2 \subset \mathbb{R}^2\) the law of the orthogonal projection (Proj\(_{S_1}\)\(v(t), \)Proj\(_{S_2}\)\(w(t)\) of \((v(t), w(t))\) is absolutely continuous with respect to the Lebesgue measure on \(S_1 \times S_2\) (cf. [29, p. 1751]).

Furthermore, due to Corollary 2.2 and [30, Corollary 2.1.2], if we could show that
\[
(\det \gamma)^{-1} \in \bigcap_{p>1} L^p(\Omega),
\]
then this implies that for any finite-dimensional subspaces \(S_1, S_2 \subset \mathbb{R}^2\) the law of the orthogonal projection (Proj\(_{S_1}\)\(v, \)Proj\(_{S_2}\)\(w\)) of \((v, w)\) has \(C^\infty\) density [29, p. 1760]. In the case of the 2D NSE, Mattingly and Pardoux proved this additional condition through [29, Theorem 6.2]; we were unable to follow the same approach for the MPF system due to the lack of symmetry in the \(w\)-equation, which is needed to write the analogous formulations [29, (3.3), (3.4)]. In short, although we may write
\[
(-((Kv \cdot \nabla)v)) = \sum_{j,k} \left( \frac{j \cdot k}{|j|^2} v_j \right) v_k = \frac{1}{2} \sum_{j,k} \left( \frac{j \cdot k}{|j|^2} v_j v_k + \frac{k \cdot j}{|k|^2} v_k v_j \right) = \sum_{j,k} c(j, k) v_j v_k
\]
if
\[
c(j, k) = \frac{1}{2} (j \cdot k) \left( \frac{1}{|j|^2} - \frac{1}{|k|^2} \right),
\]
it seems difficult to take advantage of such a symmetrization technique in (2.3b) due to \(-((Kv \cdot \nabla)w). We also mention that the proof of [22, Corollary 4.2] concerning the NSE also relies on [29, Theorem 6.2], and hence we could not extend it to the MPF system.

3 Proof

For simplicity, we write \(A \leq_a B\) and \(A \approx_a B\) if there exists a constant \(C = C(a, b) \geq 0\) such that \(A \leq CB\) and \(A = CB\), respectively. Throughout these proofs, the dependence of parameters is an important issue. We set for \(a \geq 0\),
\[
e_a^x \triangleq \sum_{k \in \mathbb{Z}^d_+} |k|^{2a} |\sigma_k^x|^2, \quad \epsilon_a^w \triangleq \sum_{k \in \mathbb{Z}^d_+} |k|^{2a} |\sigma_k^w|^2, \quad \alpha_{\text{max}}^x \triangleq \max_{k \in \mathbb{Z}^d_+} |\sigma_k^x|, \quad \alpha_{\text{max}}^w \triangleq \max_{k \in \mathbb{Z}^d_+} |\sigma_k^w|.
\]
Proposition 3.1 (cf. [29, Corollary A.2], [37, Lemma 3.2.1]). Under the hypothesis of Theorem 2.1, there exist \( \eta_0 = \eta_0(\mu, \chi, y, \sigma_{\max}^w, \sigma_{\max}^v) > 0 \) and \( \beta = \beta(\mu, \chi, y) > 0 \) such that for all \( \eta \in (0, \eta_0) \),

\[
E[\exp(\eta \sup_{t \in [0, T]} (\|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2))] \leq C_0 E[\exp(\beta \eta(\|v(0)\|_{L^2}^2 + \|w(0)\|_{L^2}^2))]
\]

and

\[
E \left[ \exp \left( \eta \int_0^T \|v\|_{L^2}^2 \, dt \right) \right] \leq C_0 E \left[ \exp(\beta \eta(\|v(0)\|_{L^2}^2 + \|w(0)\|_{L^2}^2)) \right],
\]

where \( C_0 = C_0(\eta, c_0^w, c_0^v, \chi, y, \mu, T) > 0 \).

Proof. We first fix \( \varepsilon \in (0, \frac{1}{7}] \). By hypothesis, \( \chi^2/(2\mu + \chi) < y \), so that for any \( \varepsilon \in (0, \frac{1}{7}] \),

\[
\frac{\chi}{(1 - \varepsilon)y} < \frac{8(1 - \varepsilon)(\mu + \chi) - 2\chi}{\chi},
\]

and hence we may select

\[
\overline{c} \in \left( \frac{\chi}{(1 - \varepsilon)y}, \frac{8(1 - \varepsilon)(\mu + \chi) - 2\chi}{\chi} \right).
\]

On (3.3a)–(3.3b), we apply Ito’s formula with \( f(t, x) = x^2 \) and integrate over \( T^2 \) to obtain

\[
d\|v\|_{L^2}^2 = [-2(\mu + \chi)\|v\|_{L^2}^2 - 2\chi(v, \Delta w)] \, dt + (2v, dN_v) + c_0^v \, dt,
\]

\[
d\|w\|_{L^2}^2 = [-4\chi\|w\|_{L^2}^2 - 2\chi(w, v)] \, dt + (2w, dN_w) + c_0^w \, dt,
\]

where we used (2.2) and (3.1). Integrating over \([0, t]\) and letting

\[
E_1(t) \triangleq \|v(t)\|_{L^2}^2 + \overline{c}\|w(t)\|_{L^2}^2 + 2\varepsilon \left( \int_0^t (\mu + \chi)\|v\|_{L^2}^2 + \overline{c}\|w\|_{L^2}^2 \, dt \right)
\]

implies

\[
E_1(t) = -2(1 - \varepsilon) \int_0^t (\mu + \chi)\|v\|_{L^2}^2 + \overline{c}\|w\|_{L^2}^2 \, dt - \int_0^t 4\chi\overline{c}\|w\|_{L^2}^2 \, dt + \|v(0)\|_{L^2}^2 + \overline{c}\|w(0)\|_{L^2}^2
\]

\[
- \int_0^t 2\chi(v, \Delta w) - 2\chi\overline{c}(w, v) \, dt + \int_0^t 2(v, dN_v) + \int_0^t 2\overline{c}(w, dN_w) + (c_0^v + \overline{c}c_0^w) t,
\]

where we estimate

\[
- 2\chi(v, \Delta w) + 2\chi\overline{c}(w, v) \leq \left( \frac{X}{2} + \frac{X\overline{c}}{4} \right)\|v\|_{L^2}^2 + 2\chi\|w\|_{L^2}^2 + 4\chi\overline{c}\|w\|_{L^2}^2
\]

(3.7)

by Young’s and Poincaré’s inequalities. This implies

\[
-2(1 - \varepsilon)(\mu + \chi) \int_0^t \|v\|_{L^2}^2 \, dt - 2(1 - \varepsilon)\overline{c}\int_0^t \|w\|_{L^2}^2 \, dt - 4\chi\overline{c}\int_0^t \|w\|_{L^2}^2 \, dt - \int_0^t 2\chi(v, \Delta w) - 2\chi\overline{c}(w, v) \, dt
\]

\[
\leq \left[ -2(1 - \varepsilon)(\mu + \chi) + \frac{X}{2} + \frac{X\overline{c}}{4} \right] \int_0^t \|v\|_{L^2}^2 \, dt + \left[ -2(1 - \varepsilon)\overline{c} + 2\chi \right] \int_0^t \|w\|_{L^2}^2 \, dt
\]

(3.8)

by Poincaré’s inequality and the choice of \( \overline{c} \) in (3.4). We let

\[
N_1(t) \triangleq \int_0^t 2(v, dN_v) + \int_0^t 2\overline{c}(w, dN_w),
\]

(3.9)
with which we may compute

\[
\langle \langle N_1, N_1 \rangle \rangle_t \leq 4(\sigma_{\text{max}}^w)^2 \int_0^t \|v\|^2_{L^2} \, d\tau + 4\bar{c}^2(\sigma_{\text{max}}^w)^2 \int_0^t \|w\|^2_{L^2} \, d\tau
\]

(3.10)

by (2.4), (2.2), the orthogonality of \(e_k^0\) and \(\tilde{e}_k^0\), and (3.2). Due to (3.4) we may choose

\[
\lambda \in \left(0, \min \left\{ \frac{2(1-\epsilon)(\mu + \chi) - \frac{\chi}{2} + \frac{X}{4}}{2(\sigma_{\text{max}}^w)^2}, \frac{2(1-\epsilon)\bar{c}y - 2\chi}{2\bar{c}^2(\sigma_{\text{max}}^w)^2} \right\} \right),
\]

so that multiplying (3.10) by \(-\frac{1}{2}\) gives

\[-\frac{\lambda}{2} \langle \langle N_1, N_1 \rangle \rangle_t \geq -2\lambda(\sigma_{\text{max}}^w)^2 \int_0^t \|v\|^2_{L^2} \, d\tau - 2\bar{c}^2(\sigma_{\text{max}}^w)^2 \int_0^t \|w\|^2_{L^2} \, d\tau.
\]

(3.11)

This leads to

\[
P\left( \sup_{t \in [0, T]} \left( N_1(t) + \left[ -2(1-\epsilon)(\mu + \chi) + \frac{X}{2} + \frac{\bar{c}^2 X}{4} \right] \int_0^t \|v\|^2_{L^2} \, d\tau + \left[ -2(1-\epsilon)\bar{c}y + 2\chi \right] \int_0^t \|w\|^2_{L^2} \, d\tau > k \right) \leq e^{-\lambda k}
\]

(3.12)

by (3.11) and the exponential martingale inequality (see, e.g., [11, p. 105]). Using our definition of \(N_1(t)\) in (3.9), we have shown

\[
E_1(t) = \|v(0)\|^2_{L^2} - \|v\|^2_{L^2} - \epsilon \|e_0^v + \bar{c}e_0^w\| t
\]

\[
\leq N_1(t) + \left[ -2(1-\epsilon)(\mu + \chi) + \frac{X}{2} + \frac{\bar{c}^2 X}{4} \right] \int_0^t \|v\|^2_{L^2} \, d\tau + \left[ -2(1-\epsilon)\bar{c}y + 2\chi \right] \int_0^t \|w\|^2_{L^2} \, d\tau
\]

by (3.6) and (3.8). Hence taking the supremum over \(t \in [0, T]\) on the right-hand side and then on the left-hand side leads to

\[
P\left( \sup_{t \in [0, T]} \left( \|v\|^2_{L^2} + \|w\|^2_{L^2} \right)(t) + 2\epsilon \left( \int_0^t (\mu + \chi)\|v\|^2_{H^1} + \bar{c}y\|w\|^2_{H^1} \, d\tau \right)
\]

\[
- \left. \|v(0)\|^2_{L^2} - \|v\|^2_{L^2} - \epsilon \|e_0^v + \bar{c}e_0^w\| t > k \right) \leq e^{-\lambda k}
\]

(3.13)

by the definition of \(E_1(t)\) and (3.12). Now if a random variable \(X\) satisfies \(P(|X| \leq C) \leq 1/C^2\) for all \(C > 0\), then \(E[X] \leq 2\). Using this fact and (3.13) leads to

\[
E\left[ e^{2 \sup_{t \in [0, T]} (\|v\|^2_{L^2} + \|w\|^2_{L^2}(t))} \right] \leq C_0 E\left[ e^{2 \|v(0)\|^2_{L^2} + \|w(0)\|^2_{L^2}} \right]
\]

if we choose

\[
C_0 = 2e^{\frac{1}{2}(\epsilon^* + \bar{c}^* \epsilon^*_0)^T}.
\]

Thus, we may claim that this holds for all \(\eta \in (0, \eta_0), \eta_0 = \frac{1}{2}\) and \(C_0 = C_0(\eta_0, \epsilon^*_0, \bar{c}^*, \chi, \gamma, M, T) > 0\). Analogous bound for (3.3) may be obtained by an identical procedure. Finally, the constant \(\beta = \beta(\mu, \chi, y) > 0\) follows from the definition of \(\bar{c}\) in (3.4). The proof of Proposition 3.1 is complete. \(\square\)
Proposition 3.2 (cf. [22, Lemma 4.10, (4.7)], [37, Lemma 3.2.2]). Under the hypothesis of Theorem 2.1, there exist constants \( \eta_0 = \eta_0(\mu, \chi, \gamma, \sigma_{\max}^w, \sigma_{\max}^v) > 0 \) and \( M > 0 \), such that for all \( \eta \in (0, \eta_0) \) and any \( M \in \mathbb{N} \),
\[
E\left[ \exp\left( \eta \sum_{n=0}^{M} \|v_n\|_{L_2}^2 + \bar{\tau} \|w_n\|_{L_2}^2 - \theta M \right) \right] \leq C_0 E\left[ \exp(\beta \eta (\|v(0)\|_{L_2}^2 + \bar{\tau} \|w(0)\|_{L_2}^2)) \right],
\]
where \( C_0 = C_0(\mu, \chi, \gamma, \sigma_{\max}^w, \sigma_{\max}^v, \epsilon^v_0, \epsilon^w_0) \).

Proof. By hypothesis,
\[
\frac{\lambda^2}{2(\mu + \frac{1}{2})} < \gamma,
\]
which certainly implies
\[
\frac{\lambda^2}{8\mu + 6\chi} < \gamma.
\]
Therefore, we may take \( \lambda \in (0, 2\gamma) \) sufficiently small so that
\[
\frac{2\lambda}{2\gamma - \lambda} < \frac{8\mu}{X} + 6 - \frac{4\lambda}{X} = \frac{8(\mu + \chi)}{X} - 2 - \frac{4\lambda}{X},
\]
and thus we may take
\[
\bar{\tau} \in \left( \frac{2\lambda}{2\gamma - \lambda}, \frac{8(\mu + \chi)}{X} - 2 - \frac{4\lambda}{X} \right).
\]
By (3.5a)–(3.5b), we have
\[
d(\|v\|_{L_2}^2 + \bar{\tau} \|w\|_{L_2}^2) = \left[ -2(\mu + \chi)\|v\|_{E_{1t}}^2 - 4\bar{\tau} \chi \|w\|_{E_{1t}}^2 - 2\bar{\tau} y \|w\|_{E_{1t}}^2 - 2\chi(\nu, \Delta w) + 2\bar{\tau} (w, v) + \epsilon^v_0 + \bar{\tau} \epsilon^w_0 \right] dt
\]
\[+ (2\nu, dN_v) + \bar{\tau} (2w, dN_w).
\]
Now we apply Ito’s formula to (3.16) with \( f(t, x) = e^{\lambda t} x \), integrate over time \([0, t]\) and define
\[
E_2(t) \equiv (\|v\|_{L_2}^2 + \bar{\tau} \|w\|_{L_2}^2)(t)
\]
to deduce
\[
E_2(t) \leq e^{-\lambda t} E_2(0) + \int_0^t e^{-\lambda (t-r)} \left[ -2(\mu + \chi)\|v\|_{E_{1t}}^2 - 4\bar{\tau} \chi \|w\|_{E_{1t}}^2 - 2\bar{\tau} y \|w\|_{E_{1t}}^2
\]
\[- 2\chi(\nu, \Delta w) + 2\bar{\tau} (w, v) \right] dr + \int_0^t e^{-\lambda (t-r)} \left( \epsilon^v_0 + \bar{\tau} \epsilon^w_0 \right) \left( 2\nu, dN_v \right) + \int_0^t \bar{\tau} e^{-\lambda (t-r)} (2w, dN_w).
\]
We use estimate (3.7) on (3.17) and apply Poincaré’s inequality, which is justified due to (3.15), to deduce
\[
E_2(t) \leq e^{-\lambda t} E_2(0) + \int_0^t e^{-\lambda (t-r)} \left[ -2(\mu + \chi) + \frac{\bar{\tau} \chi}{2} + \lambda \right] \|v\|_{E_{1t}}^2 \ dr + \int_0^t e^{-\lambda (t-r)} \left[ -2\bar{\tau} y + 2\bar{\tau} + \lambda \bar{\tau} \right] \|w\|_{E_{1t}}^2 \ dr
\]
\[+ \left( \epsilon^v_0 + \bar{\tau} \epsilon^w_0 \right) \left( 2\nu, dN_v \right) + \int_0^t \bar{\tau} e^{-\lambda (t-r)} (2w, dN_w).
\]
We let
\[
N_2(t) \equiv \int_0^t e^{-\lambda (t-r)} (2\nu, dN_v) + \int_0^t e^{-\lambda (t-r)} (2\bar{\tau} w, dN_w),
\]
so that for \( \rho > 0 \) sufficiently small such that
\[
0 < \rho < \min\left( \frac{2(\mu + \chi) - \frac{\chi}{2} - \lambda}{2(\sigma_{\max}^w)^2}, \frac{2\bar{\tau} y - 2\chi - \lambda \bar{\tau}}{2\bar{\tau}^2 (\sigma_{\max}^v)^2} \right)
\]
we may compute
\[ \frac{\rho}{2} \langle \langle N_2, N_2 \rangle \rangle_t \leq \left[ 2(\mu + \chi) - \frac{X}{2} - \frac{\lambda^2}{4} - \lambda \right] \int_0^t e^{-\lambda(t-r)} \| v \|_{L^2}^2 \, dr + \left[ 2\tau^2 Y - 2\chi - \lambda \tau \right] \int_0^t e^{-\lambda(t-r)} \| w \|_{L^2}^2 \, dr \]
by (2.4) and (2.2). Applying this to (3.18) and using (3.19) and the exponential martingale inequality, we deduce
\[ P \left( \left\{ E_2(t) - e^{-\lambda t} E_2(0) - \frac{e^\lambda}{\lambda} \left( e^{\lambda t} \tau^2 Y - \lambda \tau \right) > k \right\} \right) \leq e^{-\lambda k}. \]
Therefore, taking \( \eta_0 \leq \frac{\rho}{2} \), for all \( \eta \in (0, \eta_0) \) we have
\[ E[\exp(\eta E_2(t))] \leq C_0 E[\exp(\eta e^{-\lambda t} E_2(0))] \]
if
\[ C_0 = 2 e^{\frac{\rho}{2}(\frac{\tau^2 Y}{4})}. \]
Taking \( \eta_0 \) even smaller if necessary, we may assume
\[ \eta_0 < \min \left\{ \frac{\rho}{\| v(0) \|_{L^2}^2 + \tau^2 \| w(0) \|_{L^2}^2}, \left( \frac{\rho}{2} \right)(1 - e^{-\lambda}) \right\}. \]
Now for the fixed \( M \in \mathbb{N} \), taking a conditional expectation with respect to \( \mathcal{F}_{M-1} \) leads to
\[ E \left[ \exp \left( \eta \sum_{n=0}^M \| v(n) \|_{L^2}^2 + \tau \| w(n) \|_{L^2}^2 \right) \right] \leq C_0^M E \left[ e^{\eta \left( \sum_{n=0}^M e^{-\lambda n} \| v(0) \|_{L^2}^2 + \tau \| w(0) \|_{L^2}^2 \right)} \right]. \]
(3.20)
We take \( \beta \) such that \( 1/(1 - e^{-\lambda}) \leq \beta \) to reach (3.14) by using (3.20). This completes the proof of Proposition 3.2.

Proposition 3.3 (cf. [29, Lemma A.3]). Under the hypothesis of Theorem 2.1, for every \( T > 0, p \in [0, \infty) \) there exists a constant
\[ C_0 = C_0 (\sigma^\nu_{\text{max}}, \sigma^w_{\text{max}}, E[\| v(0) \|_{L^2}^2], E[\| w(0) \|_{L^2}^2], c^\nu_0, c^w_0, T, \chi, \eta, \mu, p) > 0 \]
such that
\[ E \left[ \sup_{t \in [0, T]} \left( \| v(t) \|_{L^2}^2 + \| w(t) \|_{L^2}^2 + \tau \| \Lambda v(t) \|_{L^2}^2 + \tau \| \Lambda w(t) \|_{L^2}^2 \right)^p \right] \leq C_0. \]
(3.21)

Proof. We fix
\[ \tau > \max \left\{ \frac{X}{Y}, \frac{2}{3} \left( \frac{\chi \mu}{Y} \right), 1 \right\} \]
(3.22)
and apply \( \Lambda \) to (2.3a)–(2.3b) to obtain
\[ d\Lambda v = [-\Lambda ((\chi \nu) \cdot \nabla v) + (\mu + \chi) \Delta \Lambda v - \chi \Delta w] \, dt + d\Lambda N_v, \]
\[ d\Lambda w = [-\Lambda ((\chi \nu) \cdot \nabla w) - 2\chi \nabla w + y \Delta w + \chi \Lambda v] \, dt + d\Lambda N_w. \]
Hence, applying Ito’s formula with \( f(t, x) = x^2 \) and integrating over \( \mathbb{T}^2 \) give
\[ d\| \Lambda v \|_{L^2}^2 = [-2(\mu + \chi) \| \Lambda v \|_{L^2}^2 - 2(\chi (\nu) \cdot \nabla \nu, \Delta \Lambda v) - 2\chi \Lambda v, \Delta w + c_v^\nu] \, dt + 2(\Lambda v, d\Lambda N_v), \]
\[ d\| \Lambda w \|_{L^2}^2 = [-4\chi \| \Lambda w \|_{L^2}^2 - 2\chi \| \Lambda w \|_{L^2}^2 - 2(\chi (\nu) \cdot \nabla w, \Delta w) + 2\chi (\Lambda w, \Lambda v) + c_w^\nu] \, dt + 2(\Lambda w, d\Lambda N_w), \]
where we remark that \( \| \nabla \|_{L^2} = \| \Lambda f \|_{L^2} \). Next, we apply Ito’s formula with \( f(t, x) = t \mu x \) and \( f(t, x) = t y x \) to (3.23a)–(3.23b), respectively, so that defining
\[ E_3(t) \triangleq \| v(t) \|_{L^2}^2 + \| w(t) \|_{L^2}^2 + t \mu \| \Lambda v(t) \|_{L^2}^2 + \tau \tau \| \Lambda w(t) \|_{L^2}^2, \]
we obtain by (3.5a)–(3.5b),
\[ E_3(t) - e_0^t t - \frac{\mu}{2} e_t^1 t^2 - \tau e_0^w t - \tau Y e_0^w t^2 \]
\[ = \|v(0)\|^2_{E^2} + \overline{c}\|w(0)\|^2_{E^2} + \int_0^t \left[-2(\mu + \chi)\|v\|^2_{E^2} - 2\chi(v, \Delta w)\right] d\tau + \int_0^t \left(2\nu(dN_v) + \mu \right) \|\Lambda v\|^2_{E^2} d\tau \]
\[ + \int_0^t \tau \mu \left[-2(\mu + \chi)\|\Delta v\|^2_{E^2} - 2(\chi v \cdot \nabla v, \Delta v) - 2\chi(\Delta v, \Delta w)\right] d\tau - \int_0^t 2\nu(\nu, d\Delta N_v) \]
\[ - \int_0^t 4\chi\overline{c}\|w\|^2_{E^2} + 2\chi\overline{c}\|w\|^2_{E^1} d\tau + \int_0^t 2\overline{c}(w, w) d\tau + \int_0^t \gamma\|\Lambda w\|^2_{E^2} d\tau \]
\[ + \int_0^t \tau \gamma \left[-4\chi\|\Lambda w\|^2_{E^2} - 2\gamma\|\Delta w\|^2_{E^2} - 2(\chi v \cdot \nabla w, \Delta w) + 2\chi(\Lambda w, \Lambda v)\right] d\tau - \int_0^t \gamma 2\gamma(w, d\Delta N_w). \]

We define
\[ M^v_t = \int_0^t 2\nu((1 - \tau \mu \Delta)dN_v), \quad M^w_t = \int_0^t 2\overline{c}(w, (1 - \tau \gamma \Delta)dN_w). \] (3.24)

Then we may write
\[ E_3(t) - e_0^t t - \frac{\mu}{2} e_t^1 t^2 - \tau e_0^w t - \tau Y e_0^w t^2 \]
\[ = \|v(0)\|^2_{E^2} + \overline{c}\|w(0)\|^2_{E^2} + \int_0^t \left[-2(\mu + \chi)\|v\|^2_{E^2} - 2\chi(v, \Delta w)\right] d\tau + M^v_t + M^w_t \]
\[ + \int_0^t \left[-4\chi\overline{c}\|w\|^2_{E^2} - \gamma\overline{c}\|w\|^2_{E^1} + 2\chi\overline{c}(w, v)\right] d\tau \]
\[ + \int_0^t \tau \mu \left[-2(\mu + \chi)\|\Delta v\|^2_{E^2} - 2(\chi v \cdot \nabla v, \Delta v) - 2\chi(\Delta v, \Delta w)\right] d\tau \]
\[ + \int_0^t \tau \gamma \left[-4\chi\|\Lambda w\|^2_{E^2} - 2\gamma\|\Delta w\|^2_{E^2} - 2(\chi v \cdot \nabla w, \Delta w) + 2\chi(\Lambda w, \Lambda v)\right] d\tau. \] (3.25)

We estimate by Young’s inequality
\[ - 2\chi(v, \Delta w) \leq \chi\|v\|^2_{E^1} + \chi\|v\|^2_{E^2}, \quad 2\chi\overline{c}(w, v) \leq \chi\overline{c}\|w\|^2_{E^2} + \chi\overline{c}\|v\|^2_{E^2}. \] (3.26)

Next, we compute by integrating by parts, Hölder’s, Gagliardo–Nirenberg’s and Young’s inequalities
\[ -2((\chi v) \cdot \nabla v, \Delta v) = 2 \int (\nabla^2 \overline{c}(v) \cdot \nabla v \cdot \nabla v \]
\[ \leq \|v\|^2_{E^2} \|\nabla v\|_{E^1} \|\Delta v\|_{E^2}. \]
\[ \leq \mu \|\Delta v\|^2_{E^2} + c\|v\|^2_{E^2} \|\nabla v\|^2_{E^1}, \] (3.27)

\[ -2((\chi v) \cdot \nabla w, \Delta w) = 2 \int (\nabla^2 \overline{c}(v) \cdot \nabla w \cdot \nabla w \]
\[ \leq \|v\|^2_{E^2} \|\nabla w\|_{E^1} \|\Delta w\|_{E^2}. \]
\[ \leq \frac{Y}{2} \|\Delta w\|^2_{E^2} + c\|v\|^2_{E^2} \|\nabla w\|^2_{E^1}. \] (3.28)

Finally, by Young’s inequality,
\[ - 2\chi(\Delta v, \Delta w) \leq \chi\|\Delta v\|^2_{E^2} + \chi\|\Delta w\|^2_{E^2}, \quad 2\chi(\Lambda w, \Lambda v) \leq \chi\|\Delta w\|^2_{E^2} + \chi\|\Lambda v\|^2_{E^2}. \] (3.29)
Applying (3.26)–(3.29) to (3.25) gives
\[ E_3(t) - \frac{\mu}{2} e_0^t t - \frac{\mu}{2} e_1^t t^2 - \tau e_{w_0}^t t - \frac{\tau Y}{2} e_{w_1}^t t^2 \]
\[ \leq \|v(0)\|_{L_2}^2 + \tau \|w(0)\|_{L_2}^2 + \int_0^t \left( \tau \|\nu\|_{L_2}^2 + \tau \nu \|\nabla v\|_{L_2}^2 + \tau \|v\|_{H_1}^2 + \tau \|\nabla v\|_{L_2}^2 \right) d\tau \]
\[ - \int_0^t (\mu + \chi) \|v\|_{H_1}^2 + \mu \left( \frac{3\mu}{2} + \chi \right) \tau \|\nabla v\|_{L_2}^2 d\tau - \int_0^t \left( \frac{\tau Y^2}{2} - \chi \mu \right) \|\Delta w\|_{L_2}^2 d\tau \]
\[ + M'_r + M''_r. \]  

(3.30)

Now we define
\[ L^r(t) \equiv - \int_0^t (\mu + \chi) \|v\|_{H_1}^2 + \mu \left( \frac{3\mu}{2} + \chi \right) \tau \|\nabla v\|_{L_2}^2 d\tau + M'_r, \]  

(3.31a)

\[ L^w(t) \equiv - \int_0^t (\tau Y^2 - \chi \mu) \|\Delta w\|_{L_2}^2 d\tau + M''_r, \]  

(3.31b)

so that, from (3.30), we have
\[ E_3(t) - \frac{\mu}{2} e_0^t t - \frac{\mu}{2} e_1^t t^2 - \tau e_{w_0}^t t - \frac{\tau Y}{2} e_{w_1}^t t^2 \leq \|v(0)\|_{L_2}^2 + \tau \|w(0)\|_{L_2}^2 + \int_0^t \left( \tau \|\nu\|_{L_2}^2 + \tau \nu \|\nabla v\|_{L_2}^2 + \tau \|v\|_{H_1}^2 + \tau \|\nabla v\|_{L_2}^2 \right) d\tau \]
\[ + \tau \|v\|_{H_1}^2 + \mu \left( \frac{3\mu}{2} + \chi \right) \tau \|\nabla v\|_{L_2}^2 d\tau - \int_0^t \left( \frac{\tau Y^2}{2} - \chi \mu \right) \|\Delta w\|_{L_2}^2 d\tau \]
\[ + L^r(t) + L^w(t). \]  

(3.32)

By our choice of \( \overline{c} \) in (3.22), we may choose \( \lambda > 0 \) such that
\[ \lambda \leq \min \left\{ \frac{\mu + \chi}{2(\sigma_{\max}^2)^2}, \frac{\left( \frac{3\mu}{2} + \chi \right)}{2(\sigma_{\max}^2)^2 T \mu}, \frac{\left( \frac{\tau Y^2}{2} - \chi \mu \right)}{2(\sigma_{\max}^2)^2 T Y^2}, \right\} \]

so that by (3.24),
\[ - \int_0^t (\mu + \chi) \|v\|_{L_2}^2 + \mu \left( \frac{3\mu}{2} + \chi \right) \tau \|\nabla v\|_{L_2}^2 d\tau \leq \frac{\lambda}{2} \left( \langle M'_r, M'_r \rangle \right), \]
\[ - \int_0^t (\tau Y - \chi \mu) \|\Delta w\|_{L_2}^2 d\tau \leq \frac{\lambda}{2} \left( \langle M''_r, M''_r \rangle \right). \]

Adding \( M'_r, M''_r \) to both sides, using (3.31a), (3.31b) and taking the supremum over \( t \in [0, T] \) on the right-hand and then the left-hand sides gives for any \( \beta \geq 0 \),
\[ \mathbb{P} \left( \sup_{t \in [0, T]} L^r(t) > \beta \right) \leq e^{-\beta \beta}, \quad \mathbb{P} \left( \sup_{t \in [0, T]} L^w(t) > \beta \right) \leq e^{-\beta \beta} \]
due to the exponential martingale inequality. Thus, for all \( p \in [1, \infty) \), we may bound both of
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| L^r(t) \right|^p \right] \quad \text{and} \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \left| L^w(t) \right|^p \right]. \]

Upon taking the supremum over \( \tau \in [0, T] \) on the right-hand and then the left-hand sides of (3.32), raising to the power of \( p \) and taking expected values, we obtain
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| E_3(t) \right|^p \right] \leq \mathbb{E} \left[ \left| e_0^t T + \frac{\mu}{2} e_1^t T^2 + \tau e_{w_0}^t T + \frac{\tau Y}{2} e_{w_1}^t T^2 + \|v(0)\|_{L_2}^2 + \|w(0)\|_{L_2}^2 \right|^p \right] \]
\[ + \mathbb{E} \left[ \left( \int_0^t \left( \tau \|\nu\|_{L_2}^2 + \tau \|\nabla v\|_{L_2}^2 + \tau \|v\|_{H_1}^2 + \tau \|\nabla v\|_{L_2}^2 \right) d\tau \right|^p \right] \]
\[ + \mathbb{E} \left[ \left( \sup_{t \in [0, T]} L^r(t) + \sup_{t \in [0, T]} \left| L^w(t) \right|^p \right) \right], \]  

(3.33)
where we know we can bound the last term. Moreover, by Proposition 3.1, we may obtain the following bound:

\[
E\left[ \left( \int_0^T \chi \|v\|_{L^2}^2 + \tau \mu c \|v\|_{E_{\text{H}}}^2 \|v\|_{L^2}^2 + \tau \gamma c \|v\|_{L^2}^2 \|w\|_{E_{\text{H}}}^2 + \tau \gamma \chi \|\Lambda v\|_{L^2}^2 \,dt \right)^p \right]
\]

\[
\leq E\left[ \sup_{t \in [0,T]} \|v(t)\|_{L^2}^{2p} \right] \left( E\left[ \sup_{t \in [0,T]} \|v(t)\|_{E_{\text{H}}}^{2p} \right] \right)^{\frac{1}{2}} \left( \int_0^T \|w\|_{E_{\text{H}}}^2 \,dt \right)^{\frac{p}{2}} \left( \int_0^T \|\Lambda v\|_{L^2}^2 \,dt \right)^{\frac{p}{2}} + E\left[ \int_0^T \|\|\Lambda v\|_{L^2}^2 \,dt \right]^{\frac{p}{2}}
\]

\[
\leq C(\beta, \eta, E[\|v(0)\|_{L^2}^2], E[\|w(0)\|_{L^2}^2], \epsilon_0', \epsilon_0'', T, \chi, \gamma, \mu, \beta).
\]

Therefore, applying this to (3.33) completes the proof of Proposition 3.3.

**Proof of Theorem 2.1.** We consider first the case \( \lambda = 0 \). Fix \( T_0 > 0, T \leq T_0, \eta > 0, s \geq 0 \) and \( \phi \in L^2 \). We let

\[
v_r(s) \equiv v(s + r), \quad w_r(s) \equiv w(s + r), \quad J_r^s \equiv J_{s,s+r}^r, \quad J_r^w \equiv J_{s,s+r}^w, \quad \phi_r \equiv \phi_{s,s+r} \quad \text{and}
\]

\[
R_r \equiv \|J_r^s\|_{E_{\text{H}}}^2 + \|J_r^w\|_{E_{\text{H}}}^2 + \theta \|J_r^\delta\|_{E_{\text{H}}}^2 + \theta \|J_r^\delta\|_{E_{\text{H}}}^2,
\]

so that taking \( t = s + r \) in (2.5a)–(2.5c) gives

\[
\partial_t J_r^s = (\mu + \chi) \Delta J_r^s - \langle (\nabla v_r) \cdot \nabla J_r^s - (\nabla J_r^s) \cdot \nabla v_r - \chi J_r^w/n^w, \quad J_r^w = J_r^w
\]

Taking \( L^2 \)-inner products of (3.35a)–(3.35b) with \( (J_r^s, J_r^w) \) and then \( (-\Delta J_r^s, -\Delta J_r^w) \), respectively, gives

\[
\frac{1}{2} \partial_t \|J_r^s\|_{E_{\text{H}}}^2 + (\mu + \chi) \|J_r^s\|_{E_{\text{H}}}^2 - \langle (\nabla v_r) \cdot \nabla J_r^s, J_r^s \rangle - \chi \langle \Delta J_r^s, J_r^s \rangle,
\]

\[
\frac{1}{2} \partial_t \|J_r^w\|_{E_{\text{H}}}^2 + 2\chi \|J_r^w\|_{E_{\text{H}}}^2 + \theta \|J_r^\delta\|_{E_{\text{H}}}^2 - \langle (\nabla w_r) \cdot \nabla J_r^w, J_r^w \rangle + \chi \langle J_r^w, J_r^w \rangle,
\]

\[
\frac{1}{2} \partial_t \|J_r^\delta\|_{E_{\text{H}}}^2 + (\mu + \chi) \|J_r^\delta\|_{E_{\text{H}}}^2 = \langle (\nabla v_r) \cdot \nabla J_r^\delta, \Delta J_r^\delta \rangle + \langle (\nabla J_r^\delta) \cdot \nabla v_r, \Delta J_r^\delta \rangle + \chi \langle \Delta J_r^\delta, \Delta J_r^\delta \rangle,
\]

\[
\frac{1}{2} \partial_t \|J_r^\delta\|_{E_{\text{H}}}^2 + 2\chi \|J_r^\delta\|_{E_{\text{H}}}^2 + \theta \|J_r^\delta\|_{E_{\text{H}}}^2 = \langle (\nabla w_r) \cdot \nabla J_r^\delta, \Delta J_r^\delta \rangle + \langle (\nabla J_r^\delta) \cdot \nabla w_r, \Delta J_r^\delta \rangle - \chi \langle J_r^\delta, \Delta J_r^\delta \rangle.
\]

Thus, by (3.34),

\[
\partial_t \xi_r = -2(\mu + \chi) \|J_r^s\|_{E_{\text{H}}}^2 - 2\langle (\nabla J_r^s) \cdot \nabla v_r, J_r^s \rangle - 2\chi \langle \Delta J_r^w, J_r^s \rangle - 2\chi \|J_r^s\|_{E_{\text{H}}}^2 + 2\chi \langle (\nabla J_r^\delta) \cdot \nabla v_r, J_r^\delta \rangle + 2\chi \langle (\nabla J_r^\delta) \cdot \nabla w_r, \Delta J_r^\delta \rangle + 2\theta \|J_r^\delta\|_{E_{\text{H}}}^2 - 2\theta \langle (\nabla J_r^\delta) \cdot \nabla J_r^\delta \rangle + 2\theta \|J_r^\delta\|_{E_{\text{H}}}^2 + 2\theta \langle (\nabla J_r^\delta) \cdot \nabla J_r^\delta \rangle + 2\theta \langle (\nabla J_r^w) \cdot \Delta J_r^\delta \rangle,
\]

We estimate for \( \delta > 0 \) and \( \theta > 0 \) to be chosen subsequently,

\[
-2\langle (\nabla J_r^s) \cdot \nabla v_r, J_r^s \rangle \leq \|J_r^s\|_{E_{\text{H}}}^2 \|J_r^s\|_{E_{\text{H}}}^2 \|v_r\|_{E_{\text{H}}}^2 \leq \delta \|J_r^s\|_{E_{\text{H}}}^2 + \frac{\epsilon_0'}{\delta} \|J_r^s\|_{E_{\text{H}}}^2 \|v_r\|_{E_{\text{H}}}^2 \leq \delta \|J_r^s\|_{E_{\text{H}}}^2 + c\xi (\theta \|v_r\|_{E_{\text{H}}}^2 + \frac{1}{\theta^2\|J_r^s\|_{E_{\text{H}}}^2})
\]

by [5, Proposition 6.1] (or equivalently [29, Lemma D.1, (D.1), p. 1785] with \( \alpha_1 = \frac{1}{2}, \alpha_2 = \alpha_3 = 0 \), interpolation and Young’s inequalities and (3.34). Next,

\[
-2\chi \langle \Delta J_r^w, J_r^s \rangle \leq \chi \|J_r^w\|_{E_{\text{H}}}^2 + \chi \|J_r^s\|_{E_{\text{H}}}^2.
\]
by Young’s inequality. Next,
\[
-2\langle (\mathcal{K}J_r^\nu) \cdot \nabla w_r, J_r^\nu \rangle \leq \|J_r^\nu\|_{\mathcal{E}^1}^2 + \frac{\delta}{\delta^2} \|J_r^\nu\|_{\mathcal{E}^1}^2 + \frac{c}{\delta} \|J_r^\nu\|_{\mathcal{E}^1}^2 \|w_r\|_{\mathcal{E}^1}^2
\]
\[
\leq \delta \|J_r^\nu\|_{\mathcal{E}^1}^2 + \frac{c}{\delta} \|J_r^\nu\|_{\mathcal{E}^1}^2 \|w_r\|_{\mathcal{E}^1}^2 + \frac{\eta}{\delta} \|w_r\|_{\mathcal{E}^1}^2 + \frac{1}{\delta (\delta^2)}
\]
\[
(3.40)
\]

by [5, Proposition 6.1], interpolation and Young’s inequalities and (3.34). Next,
\[
2\chi(J_r^\nu, J_r^\nu) \leq \chi_{\delta}
\]
(3.41)

by Young’s inequality and (3.34). The next estimate must be done differently with a bit more care:
\[
2\theta\langle (\mathcal{K}v_r) \cdot \nabla v_r, \Delta J_r^\nu \rangle \leq \theta \|v_r\|_{\mathcal{E}^1}^2 + \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2
\]
\[
\leq C_0 \theta \|v_r\|_{\mathcal{E}^1}^2 + \|v_r\|_{\mathcal{E}^1}^2 + \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{c}{\eta^2} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{\eta}{\delta} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2
\]
\[
\leq \theta \delta \|J_r^\nu\|_{\mathcal{E}^1}^2 + \frac{c}{\eta^2} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{\eta}{\delta} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{\eta}{\delta} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2
\]
(3.42)

by integration by parts, Hölder’s inequality, the embedding of $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, Poincaré’s, Gagliardo–Nirenberg’s and Young’s inequalities and (3.34). Next,
\[
2\theta\chi(\Delta J_r^\nu, J_r^\nu) \leq \theta \chi(\|\Delta J_r^\nu\|_{\mathcal{E}^1}^2, \|\Delta J_r^\nu\|_{\mathcal{L}^2}^2).
\]
(3.43)

Next,
\[
2\theta\langle (\mathcal{K}v_r) \cdot \nabla v_r, \Delta J_r^\nu \rangle \leq \theta \|v_r\|_{\mathcal{E}^1}^2 + \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2
\]
\[
\leq \theta \|v_r\|_{\mathcal{E}^1}^2 + \|v_r\|_{\mathcal{E}^1}^2 + \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{c}{\eta^2} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2
\]
\[
\leq \delta \|J_r^\nu\|_{\mathcal{E}^1}^2 + \frac{c}{\eta^2} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{\eta}{\delta} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2 + \frac{\eta}{\delta} \|v_r\|_{\mathcal{E}^1}^2 \|\nabla J_r^\nu\|_{\mathcal{L}^2}^2
\]
(3.44)

by Hölder’s inequality, the embedding of $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, Poincaré’s, Gagliardo–Nirenberg’s and Young’s inequalities and (3.34). Next,
by [5, Proposition 6.1], interpolation and Young’s inequalities and (3.34). Finally,

$$- 2\theta \chi (J^{\nu}_{r}, \Delta J^{\nu}_{r}) \leq \frac{\theta \chi^2}{\delta} \zeta_{r} + \theta \delta \|\Delta J^{\nu}_{r}\|^{2}_{L^{2}}$$

(3.47)

by Young’s inequality and (3.34). Applying (3.38)–(3.47) in (3.37) gives

$$\partial_{r} \zeta_{r} \leq \left[-2\mu - \chi + 2\delta + \theta \delta \|J^{\nu}_{r}\|^{2}_{E_{t}} + \left(\left[c \theta \eta + \frac{c \theta}{\eta \delta^2} + \frac{3\eta}{4} + \frac{\theta \delta}{\eta \delta^2}\right]\|\nu\|^{2}_{E_{t}} + \left[c \theta \eta + \frac{\theta \delta}{2}\right]\|w\|^{2}_{E_{t}}\right]\zeta_{r} + \left(\frac{c}{\eta \delta^2} + \chi + \frac{\theta \chi^2}{\delta}\right)\zeta_{r} + [-2\mu + \chi + \theta \delta - 4\theta \chi \|\Delta J^{\nu}_{r}\|^{2}_{E_{t}} - 4\chi \|\nu\|^{2}_{E_{t}}]ight].$$

(3.48)

Now for \( \theta \in (0, 1) \), we take \( \delta > 0 \) sufficiently small so that

$$-2\mu - \chi + 2\delta + \theta \delta \leq 0,$$

$$-2\mu - \chi + 2\delta \leq 0,$$

$$\theta \chi - 2\theta \gamma + 3\theta \delta \leq 0,$$

$$-2\gamma + \chi + \theta \delta - 4\theta \chi < 0.$$

We fix such \( \delta > 0 \) first and then take \( \theta \in (0, 1) \) smaller to obtain

$$\partial_{r} \zeta_{r} \leq \left[C_{1} + \eta \|\nu\|^{2}_{E_{t}} + \|w\|^{2}_{E_{t}}\right]\zeta_{r},$$

where \( C_{1} = C_{1}(\theta, \chi, \eta) \). By Gronwall’s inequality, this leads to

$$\sup_{0 \leq t \leq T} \left(\|J^{\nu}_{s,t}, \phi\|^{2}_{E_{t}} + \|J^{\nu}_{s,t}, \phi\|^{2}_{E_{t}} + \theta \|J^{\nu}_{s,t}, \phi\|^{2}_{E_{t}} + \theta \|J^{\nu}_{s,t}, \phi\|^{2}_{E_{t}}\right) \leq \left(\|\phi\|^{2}_{E_{t}} + \|\phi\|^{2}_{E_{t}}\right) e^{\int_{t}^{T} C_{1} + \eta \|\nu\|^{2}_{E_{t}} + \|w\|^{2}_{E_{t}} \, dt},$$

which completes the proof of Theorem 2.1 (i) in case \( \lambda = 0 \). Taking \( \eta > 0 \) sufficiently small, Proposition 3.1 immediately completes the proof of Theorem 2.1 (ii) in case \( \lambda = 0 \).

Next, we consider the case \( \lambda = 1 \); due to similarity, we only sketch its proof. We define

$$\zeta_{r} \equiv \|J^{\nu}_{r}\|^{2}_{E_{t}} + \theta \|J^{\nu}_{r}\|^{2}_{E_{t}} + \theta \|J^{\nu}_{r}\|^{2}_{E_{t}}.$$  

(3.49)

From (3.36a)–(3.36d) we compute by (3.48),

$$\partial_{r} \zeta_{r} \leq -2(\mu + \chi)\|J^{\nu}_{r}\|^{2}_{E_{t}} - 2(\langle\Delta J^{\nu}_{r}\rangle \cdot \nu_{r}, J^{\nu}_{r}) - 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) - 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) - 4\chi\|J^{\nu}_{r}\|^{2}_{E_{t}},$$

$$- 2\chi\|J^{\nu}_{r}\|^{2}_{E_{t}} - 2\chi(\Delta J^{\nu}_{r}, \nu_{r}, J^{\nu}_{r}) + 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) + \theta \|J^{\nu}_{r}\|^{2}_{E_{t}},$$

$$+ \theta \left(-2(\mu + \chi)\|J^{\nu}_{r}\|^{2}_{E_{t}} - 2\chi(\Delta J^{\nu}_{r}, \nu_{r}, J^{\nu}_{r}) + 2\chi(\Delta J^{\nu}_{r}, \nu_{r}, J^{\nu}_{r}) + 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) + 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) + \theta \|J^{\nu}_{r}\|^{2}_{E_{t}},$$

$$+ \theta \left(-2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) - 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) + 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) + 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) - 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r})\right).$$

(3.49)

We may use the same estimate of (3.38) and (3.48) to deduce for \( \delta > 0 \) and \( \theta > 0 \) to be chosen subsequently,

$$- 2\langle\Delta J^{\nu}_{r}\rangle \cdot \nu_{r}, J^{\nu}_{r} \leq \delta\|J^{\nu}_{r}\|^{2}_{E_{t}} + \frac{1}{\delta^{2}}\|\nu_{r}\|^{2}_{E_{t}}.$$

(3.50)

Use the same estimate of (3.40), Young’s inequality and (3.48) to deduce

$$- 2\langle\Delta J^{\nu}_{r}\rangle \cdot \nu_{r}, J^{\nu}_{r} \leq \delta\|J^{\nu}_{r}\|^{2}_{E_{t}} + \frac{1}{\delta^{2}}\|\nu_{r}\|^{2}_{E_{t}}.$$

(3.51)

Next, it is straightforward to estimate by Young’s inequality and (3.48),

$$- 2\chi(\Delta J^{\nu}_{r}, J^{\nu}_{r}) \leq \chi\|J^{\nu}_{r}\|^{2}_{E_{t}} + \chi\|J^{\nu}_{r}\|^{2}_{E_{t}}.$$  

(3.52)

Next, using an identical estimate of (3.42), we deduce

$$\theta \delta\|\Delta J^{\nu}_{r}\|^{2}_{E_{t}} \leq 2\theta \delta\|\Delta J^{\nu}_{r}\|^{2}_{E_{t}} + \frac{c_{T_{0}}}{{\eta \delta^2}}\|\nu_{r}\|^{2}_{E_{t}}.$$  

(3.53)
by Poincaré’s and Young’s inequalities and (3.48). Next, by (3.43),
\[
\theta r^2 ||(\mathcal{J}^v_r') \cdot \nabla v_r, \Delta J^v_r ||_{L^2}^2 \leq r \theta \delta ||\Delta J^v_r ||_{L^2}^2 + \frac{T_0 \theta c}{\eta \delta^2} ||v_r||_{L^1}^2 \gamma_r + \frac{\eta}{8} ||v_r||_{L^1}^2 \gamma_r.
\] (3.54)

Next, by Young’s inequality
\[
2 \theta r^2 (\Delta J^v_r, \Delta J^v_r) \leq \theta r^2 (||\Delta J^v_r ||_{L^2}^2 + ||\Delta J^v_r ||_{L^2}^2).
\] (3.55)

Next, by (3.45),
\[
2 \theta r^2 (||\mathcal{J}^v_r' \cdot \nabla J^v_r', \Delta J^v_r' ||_{L^2}^2 + \frac{c \theta T_0}{\eta \delta^2} ||v_r||_{L^1}^2 \gamma_r + r \theta \delta ||\Delta J^v_r' ||_{L^2}^2 + \frac{\eta}{8} ||v_r||_{L^1}^2 \gamma_r
\] (3.56)
due to (3.48). Next, by (3.46),
\[
2 \theta r^2 (||\mathcal{J}^w_r' \cdot \nabla w_r, \Delta J^w_r' ||_{L^2}^2 + \frac{\gamma_r}{2} ||w_r||_{L^1}^2 \gamma_r + \frac{\theta c T_0}{\eta \delta^2} ||w_r||_{L^1}^2 \gamma_r
\] (3.57)
due to (3.48). Finally, by Young’s inequality and (3.48),
\[
-2 \theta r^2 (J^v_r', \Delta J^v_r') \leq \frac{\theta T_0 \chi^2}{\delta} \gamma_r + \theta r \delta ||\Delta J^v_r' ||_{L^2}^2.
\] (3.58)

Applying (3.50)–(3.58) into (3.49) gives
\[
\partial_t \gamma_r \leq [-2 \mu - \chi + 2 \delta + \theta ||J^v_r'||_{L^1}^2 + \left(\frac{c \theta \eta + \frac{c T_0 \theta}{\eta^2 \delta^2} + \eta}{2} + \frac{T_0 \theta c}{\eta \delta^2}\right) ||v_r||_{L^1}^2 \gamma_r \right] + \left(\frac{\theta c \eta + \frac{\theta c T_0}{\eta \delta^2}}{2}\right) ||w_r||_{L^1}^2 \gamma_r
\]
\[
+ \left[-2 \theta r \mu - \theta r \chi + 3 r \theta \delta ||\Delta J^v_r'||_{L^2}^2 + \theta r \chi - \theta r \delta + 3 \theta r \delta ||\Delta J^v_r'||_{L^2}^2\right].
\]

Using that \(\chi - 2 \gamma < 0\) by hypothesis, we first take \(\theta \in (0, 2 \mu + \chi)\) and then \(\delta > 0\) sufficiently small so that
\[
-2 \mu - \chi + 2 \delta + \theta \leq 0, \quad -2 \theta r \mu - \theta r \chi + 3 \theta r \delta < 0,
\]
\[
\chi - 2 \gamma + \theta - 4 \theta r \chi + r \theta \delta < 0, \quad \theta r \chi - 2 \theta r \delta + 3 \theta r \delta < 0.
\]

We fix such \(\delta > 0\) and then \(\theta \in (0, 2 \mu + \chi)\) smaller so that
\[
c \theta \eta + \frac{c T_0 \theta}{\eta^2 \delta^2} + \frac{\eta}{2} + \frac{T_0 \theta c}{\eta \delta^2} < \eta, \quad c \theta \eta + \frac{\eta}{2} + \frac{\theta c T_0}{\eta \delta^2} < \eta,
\]
so that we obtain for \(C_1 = C_1(\theta, \eta, \chi, T_0) > 0\),
\[
\partial_t \gamma_r \leq (C_1 + \eta (||v_r||_{L^1}^2 + ||w_r||_{L^1}^2)) \gamma_r.
\]

Thus, Gronwall’s inequality leads to
\[
\sup_{0 \leq t \leq T} \left(||J^v_r'||_{L^2}^2 + ||J^w_r'||_{L^2}^2 + \theta (t - s) [||J^v_r'||_{L^1}^2 + ||J^w_r'||_{L^1}^2]\right) \leq \left(||\phi||_{L^2}^2 e^{\int_0^T C_1 \eta (||v||_{L^1} + ||w||_{L^1})} dr\right),
\]
which completes the proof of Theorem 2.1 (i) in case \(\lambda = 1\). By Proposition 3.1, taking \(\eta > 0\) sufficiently small, the proof of Theorem 2.1 (ii) in case \(\lambda = 1\) is also complete. 

Proof of Corollary 2.2. We now prove Corollary 2.2. In fact, we prove a more precise and stronger statement; we derive the estimates on higher Malliavin derivatives of \((v, w)(t)\). We let \(k_1, \ldots, k_n \in \mathbb{Z}^n_r \cup \mathbb{Z}^w_s\) and \(s = (s_1, \ldots, s_n) \in \mathbb{R}^n_r\) and let
\[
\mathcal{V}_{k_1, \ldots, k_n}^n(t) \quad \text{and} \quad \mathcal{V}_{k_1, \ldots, k_n}^n(t)
\]
be the \(n\)-th Malliavin derivative of \(v(t)\) and \(w(t)\), respectively. We consider the \(n\)-linear operator \(K^a_{s, t}\) acting on \(\phi = (\phi^1, \ldots, \phi^n)\), \(\phi^i \in L^2\) for all \(i = 1, \ldots, n\), such that
\[
K^a_{s, t}(Q q_{k_1}, \ldots, Q q_{k_n}) = (\mathcal{V}_{k_1, \ldots, k_n}^n(t), \mathcal{V}_{k_1, \ldots, k_n}^n(t)).
\]
and it satisfies
\begin{align}
\partial_t K_{s,t}^{n,v} \phi &= (\mu + \chi) \Delta K_{s,t}^{n,v} \phi - B(\nabla v, K_{s,t}^{n,v} \phi) - B(\nabla v, K_{s,t}^{n,v} \phi, v) + F_{s,t}^n \phi - \chi \Delta K_{s,t}^{n,w} \phi, & t \geq \sqrt{s}, \\
\partial_t K_{s,t}^{n,w} \phi &= -2\chi K_{s,t}^{n,w} \phi + \gamma \Delta K_{s,t}^{n,w} \phi - B(\nabla v, K_{s,t}^{n,w} \phi) - B(\nabla v, K_{s,t}^{n,w} \phi, w) \\
&\quad + G_{s,t}^n \phi + H_{s,t}^n \phi + \chi K_{s,t}^{n,v} \phi, \quad t \geq \sqrt{s}, \\
(K_{s,t}^{n,v} \phi, K_{s,t}^{n,w} \phi) &= 0,
\end{align}
where
\begin{align}
F_{s,t}^n \phi &\equiv - \sum_{(a,b) \in \text{part}(n)} B(\nabla K_{s,t}^{a,v} \phi_a, K_{s,t}^{b,v} \phi_b) + B(\nabla K_{s,t}^{a,v} \phi_a, K_{s,t}^{b,w} \phi_b, K_{s,t}^{a,w} \phi_a), \\
G_{s,t}^n \phi &\equiv - \sum_{(a,b) \in \text{part}(n)} B(\nabla K_{s,t}^{a,v} \phi_a, K_{s,t}^{b,w} \phi_b), \\
H_{s,t}^n \phi &\equiv - \sum_{(a,b) \in \text{part}(n)} B(\nabla K_{s,t}^{a,v} \phi_a, K_{s,t}^{a,w} \phi_a),
\end{align}

part(n) is the set of partitions of \( \{1, \ldots, n\} \) into two sets none of which is empty,
\[
|a| = \text{card } a, \quad \phi_a = (\phi^{a_1}, \ldots, \phi^{a_{|a|}}), \quad s_a = (s_{a_1}, \ldots, s_{a_{|a|}}), \quad \bigvee s = s_{a_1} \vee \cdots \vee s_{a_{|a|}}.
\]

Corollary 2.2 follows from the following proposition. \( \square \)

**Proposition 3.4** (cf. [37, Lemma 3.2.6], [29, Lemma C.1]). For any \( \eta > 0, t > 0, n \in \mathbb{N} \) and \( p \in [1, \infty) \), there exist constants \( c = c(t, \mu, \chi, n) \) and \( \theta = \theta(t, \mu, \chi, \eta, p) \) such that
\[
\mathbb{E}\left( \sum_{k_1, \ldots, k_n \in \mathbb{R}^d \backslash \mathbb{Z}^d} \int_0^t \left( \sum_{k_1, \ldots, k_n \in \mathbb{R}^d \backslash \mathbb{Z}^d} \|v^{w,k}_{k_1,\ldots,k_n,s_k}(t)\|_{L^2}^2 + \|v^{w,k}_{k_1,\ldots,k_n,s_k}(t)\|_{L^2}^2, ds_1 \ldots ds_n \right)^p \right) \leq ce^{\eta t}\left(1+t^\eta\right)^p,
\]
so that \( (v, w)(t) \in \mathcal{D}^{\infty} \) for all \( t > 0 \).

**Proof.** Firstly,
\begin{align}
\|F_{s,t}^n \phi\|_{L^2} &\leq \sum_{(a,b) \in \text{part}(n)} \|K_{s,t}^{a,v} \phi_a\|_{L^2} \|K_{s,t}^{b,v} \phi_b\|_{L^2},
\end{align}
by (3.61), Hölder’s inequality and \( H^2 \hookrightarrow L^\infty \). Similarly by (3.61), (3.62),
\begin{align}
\|G_{s,t}^n \phi\|_{L^2} &\leq \sum_{(a,b) \in \text{part}(n)} \|K_{s,t}^{a,v} \phi_a\|_{H^1} \|K_{s,t}^{b,w} \phi_b\|_{H^1}, \\
\|H_{s,t}^n \phi\|_{L^2} &\leq \sum_{(a,b) \in \text{part}(n)} \|K_{s,t}^{b,v} \phi_b\|_{H^1} \|K_{s,t}^{a,w} \phi_a\|_{H^1}.
\end{align}

Now we may use the variation-of-constants formula on (2.5a)–(2.5c) and (3.59a)–(3.59c) to write
\[
K_{s,t}^n \phi = \int_{\sqrt{s}}^t J_{r,t} F_{s,r}^n \phi + G_{s,r}^n \phi + H_{s,r}^n \phi \ dr,
\]
with which we write
\[
\|K_{s,t}^n \phi\|_{H^1} \leq \int_{\sqrt{s}}^t \|J_{r,t} F_{s,r}^n \phi\|_{H^1} + \|J_{r,t} G_{s,r}^n \phi\|_{H^1} + \|J_{r,t} H_{s,r}^n \phi\|_{H^1} \ dr.
\]
By applying Theorem 2.1 (ii) with \( \lambda = 1 \), we obtain
\[
\|K^\|_{X_r, T}^2 \leq T_{n, \eta, \mu, \chi, \epsilon, \epsilon_0} e^{\Theta \eta |I(0)|^{2}_{L^2}} + |W(0)|^{2}_{L^2} \left( \int \frac{1}{\sqrt{s}} \, dr \right)
\times \sup_{(a, b) \in \text{part}(n)} \left( \frac{1}{2} \right) \left( \| F^{(n)}_{\tau, r} \phi \|_{L^2} + \| G^{(n)}_{\tau, r} \phi \|_{L^2} + \| H^{(n)}_{\tau, r} \phi \|_{L^2} \right)
\times \sup_{(a, b) \in \text{part}(n)} \left( \frac{1}{2} \right) \left( \| K^{(a), \nu}_{\tau, r} \phi \|_{L^2} + \| K^{(b), \nu}_{\tau, r} \phi \|_{L^2} + \| K^{(b), \nu}_{\tau, r} \phi \|_{L^2} \right),
\]
where we used (3.63)–(3.65). On the other hand, the induction base case \( n = 1 \) is done by Theorem 2.1 (2). Inductively, the proof of Proposition 3.4 is complete.

We finally prove Corollary 2.3.

**Proof of Corollary 2.3.** Let us fix \( C_0 > 0, p \in [1, \infty), T > 0 \) and \( \lambda > 0 \); for simplicity, we assume \( p \geq 8 \). By Theorem 2.1 (ii) with \( \lambda = 1, s = 0 \), we may find \( \eta > 0 \) sufficiently small so that
\[
T(\| F^p_{0, T} \|_{L^2}^2 + \| J^w_{0, T} \phi \|_{L^2}^2) \leq \left( \frac{C_2}{\theta^2} \right) e^{\Theta \eta |I(0)|^{2}_{L^2}} + |W(0)|^{2}_{L^2} + C_1 T \| \phi \|_{L^2}^2,
\]
where \( C_1 = C_1(\theta, \chi, \eta, T_0) > 0, C_2 = C_2(T_0, \eta, \mu, \chi, \epsilon, \epsilon_0) > 0 \) and \( \Theta = \Theta(\mu, \chi, \eta) > 0 \). As \( \Theta \) is independent of \( \eta > 0 \), we may take \( \eta > 0 \) even smaller if necessary to attain \( (\frac{C_2}{\theta^2}) \| \phi \|_{L^2} \). Thus, if \( \pi_t \) is the orthogonal projection onto the set of Fourier modes with \( |k| \leq M \), then we compute
\[
E[\| (1 - \pi_t)^p F^p_{0, T} \phi, (1 - \pi_t) J^w_{0, T} \phi \|_{L^2}^p] \leq \left( \frac{C_2}{\theta^2} \right) e^{\Theta \eta |I(0)|^{2}_{L^2}} + |W(0)|^{2}_{L^2} \| \phi \|_{L^2}^p
\]
by (3.66). Taking \( M > 0 \) sufficiently large so that
\[
C_0 \leq \left( \frac{C_2}{\theta^2} \right) e^{\Theta \eta |I(0)|^{2}_{L^2}} + |W(0)|^{2}_{L^2}
\]
gives the desired result (2.6).

Next, we prove the second inequality (2.7). We denote \( \zeta_0 \equiv (1 - \pi_t)^p \phi \) and set
\[
\zeta_0 \equiv f_{0, 0} \zeta_0, \quad \zeta_t \equiv (\zeta_t^r, \zeta_t^w) \equiv f_{0, t} \zeta_0 \equiv (f_{0, t}^r \zeta_0, f_{0, t}^w \zeta_0),
\]
and hence
\[
\zeta_t^r, t^r \equiv \pi_t \zeta_t^r, \quad \zeta_t^w, t^w \equiv (1 - \pi_t) \zeta_t^w, \quad \zeta_t^w, t^w \equiv \pi_t \zeta_t^w, \quad \zeta_t^w, t^w \equiv (1 - \pi_t) \zeta_t^w.
\]
Applying \( \pi_t \) and \((1 - \pi_t)^p \) to (2.5a)–(2.5c) with \( s = 0, \phi = \zeta_0 \) gives
\[
\begin{align*}
\partial_t \zeta_t^{r, l} & = (\mu + \chi) \Delta \zeta_t^{r, l} - \pi_t ((\chi \zeta_t^r \cdot \nabla) \zeta_t^r) - \pi_t ((\chi \zeta_t^w \cdot \nabla) \zeta_t^w) - \chi \Delta \zeta_t^{w, l}, \\
\partial_t \zeta_t^{w, l} & = -2 \chi \zeta_t^{w, l} + \gamma \Delta \zeta_t^{w, l} - \pi_t ((\chi \zeta_t^r \cdot \nabla) \zeta_t^w) - \pi_t ((\chi \zeta_t^w \cdot \nabla) \zeta_t^w) + \chi \Delta \zeta_t^{r, l}, \\
\zeta_t^{w, l} & = \zeta_0^{w, l} = 0,
\end{align*}
\]
and
\[
\begin{align*}
\partial_t \zeta_t^{r, h} & = (\mu + \chi) \Delta \zeta_t^{r, h} - \pi_h ((\chi \zeta_t^r \cdot \nabla) \zeta_t^r) - \pi_h ((\chi \zeta_t^w \cdot \nabla) \zeta_t^w) - \chi \Delta \zeta_t^{w, h}, \\
\partial_t \zeta_t^{w, h} & = -2 \chi \zeta_t^{w, h} + \gamma \Delta \zeta_t^{w, h} - \pi_h ((\chi \zeta_t^r \cdot \nabla) \zeta_t^w) - \pi_h ((\chi \zeta_t^w \cdot \nabla) \zeta_t^w) + \chi \Delta \zeta_t^{r, h},
\end{align*}
\]
by (3.67). Taking $L^2$-inner products on (3.68a), (3.68b) and (3.69a)–(3.69b) with $\zeta^{v,l}_t$, $\zeta^{w,l}_t$, $\zeta^{v,h}_t$, $\zeta^{w,h}_t$, respectively, and using the divergence-free property of $\mathfrak{X}v$ lead to

$$\partial_t \|\zeta^{v,l}_t\|_{L^2}^2 = -2(\mu + \chi)\|\zeta^{v,l}_t\|_{L^2}^2 - 2 \left(\mathfrak{X}(v \cdot v)\zeta^{v,l}_t, \zeta^{v,l}_t\right) - 2 \left(\mathfrak{X}\zeta^{v,l}_t \cdot v, \zeta^{v,l}_t\right) - 2X \Delta\zeta^{v,l}_t, \zeta^{v,l}_t, \tag{3.70a}\right.$$  

$$-2 \left(\mathfrak{X}\zeta^{w,l}_t \cdot v, \zeta^{w,l}_t\right) - 2X \Delta\zeta^{w,l}_t, \zeta^{w,l}_t, \tag{3.70b}\right.$$  

$$-2 \left(\mathfrak{X}\zeta^{w,h}_t \cdot v, \zeta^{w,h}_t\right) - 2X \Delta\zeta^{w,h}_t, \zeta^{w,h}_t, \tag{3.70c}\right.$$  

$$-2 \left(\mathfrak{X}\zeta^{w,h}_t \cdot v, \zeta^{w,h}_t\right) - 2X \Delta\zeta^{w,h}_t, \zeta^{w,h}_t. \tag{3.70d}\right.$$  

We estimate (3.70c), (3.70d) first as follows:

$$-2 \left(\mathfrak{X}(v \cdot v)\zeta^{v,l}_t, \zeta^{v,l}_t\right) \leq \|v\|_{H^1}^2 \|\zeta^{v,l}_t\|_{L^2}^2 + \|\zeta^{v,l}_t\|_{H^1} \|\zeta^{v,l}_t\|_{L^2} - 2 \left(\mathfrak{X}\zeta^{v,l}_t \cdot v, \zeta^{v,l}_t\right) - 2X \Delta\zeta^{v,l}_t, \zeta^{v,l}_t \leq c\|\zeta^{v,l}_t\|_{L^2}^2 + \|v\|_{H^1}^2 \|\zeta^{v,l}_t\|_{L^2}^2 \tag{3.71}\right.$$  

by integration by parts, Hölder’s inequality, the embeddings of $H^{1/2} \subset L^\infty$, $H^1 \subset L^4$ and $H^{1/2} \subset L^4$ and Poincaré’s and Young’s inequalities. Next, we estimate

$$-2X \Delta\zeta^{w,h}_t, \zeta^{w,h}_t \leq \|\zeta^{w,h}_t\|_{L^2}^2 + \|\zeta^{v,h}_t\|_{L^2}^2 \tag{3.72}\right.$$  

by Young’s inequality. Next, we estimate

$$-2 \left(\mathfrak{X}(v \cdot v)\zeta^{w,h}_t, \zeta^{w,h}_t\right) \leq \|v\|_{H^1}^2 \|\zeta^{w,h}_t\|_{L^2}^2 + \|\zeta^{w,h}_t\|_{H^1} \|\zeta^{w,h}_t\|_{L^2} - 2 \left(\mathfrak{X}\zeta^{w,h}_t \cdot v, \zeta^{w,h}_t\right) - 2X \Delta\zeta^{w,h}_t, \zeta^{w,h}_t \leq c\|\zeta^{w,h}_t\|_{L^2}^2 + \|v\|_{H^1}^2 \|\zeta^{w,h}_t\|_{L^2}^2 \tag{3.73}\right.$$  

by integration by parts, Hölder’s inequalities, the embeddings of $H^{3/2} \subset L^\infty$, $H^1 \subset L^4$ and $H^{3/2} \subset L^4$ and Poincaré’s and Young’s inequalities. Finally, we estimate

$$2X \zeta^{w,h}_t, \zeta^{w,h}_t \leq \frac{\chi}{M^2} \zeta^{w,h}_t, \zeta^{w,h}_t \tag{3.74}\right.$$  

by Young’s inequality. Applying (3.71)–(3.74) in (3.70c), (3.70d) gives

$$\partial_t\left(\|\zeta^{v,l}_t\|_{L^2}^2 + \|\zeta^{w,h}_t\|_{L^2}^2\right) \leq \left(\left(-2\mu - X + \epsilon + \frac{X}{M^2}\right)\|\zeta^{v,l}_t\|_{L^2}^2 + \left(-2\mu - X - \epsilon\right)\|\zeta^{w,h}_t\|_{L^2}^2\right) + \left(-2\mu - X + \epsilon\right)\|\zeta^{w,h}_t\|_{L^2}^2 + \left(-2\mu - X - \epsilon\right)\|\zeta^{w,h}_t\|_{L^2}^2 + \left(-2\mu - X + \epsilon\right)\|\zeta^{w,h}_t\|_{L^2}^2 + \left(-2\mu - X - \epsilon\right)\|\zeta^{w,h}_t\|_{L^2}^2.$$

For $\epsilon > 0$ sufficiently small and $M > 0$ sufficiently large, we may find $c_1 > 0$ small such that

$$\left(-2\mu - X + \epsilon + \frac{X}{M^2}\right)\|\zeta^{v,l}_t\|_{L^2}^2 + \left(-2\mu - X - \epsilon\right)\|\zeta^{w,h}_t\|_{L^2}^2 \leq \left(-c_1\|\zeta^{v,l}_t\|_{L^2}^2 + \left(-c_1\|\zeta^{w,h}_t\|_{L^2}^2 \leq -c_1M^2(\|\zeta^{v,l}_t\|_{L^2}^2 + \|\zeta^{w,h}_t\|_{L^2}^2).$$

Thus, we may find $c_2 > 0$ large so that by the interpolation inequality, we may deduce

$$\partial_t\left(\|\zeta^{v,l}_t\|_{L^2}^2 + \|\zeta^{w,h}_t\|_{L^2}^2\right) \leq e^{c_1M^2t}c_2\|\zeta^{v,l}_t\|_{L^2}^2(\|v\|_{H^1} + \|w\|_{H^1})(\|v\|_{H^1} + \|w\|_{H^1}) \leq e^{c_1M^2t}c_2\|\zeta^{v,l}_t\|_{L^2}^2(\|v\|_{H^1} + \|w\|_{H^1})(\|v\|_{H^1} + \|w\|_{H^1}).$$
so that integrating over \([0, t]\) and taking the power of \(\frac{C}{2}\) give
\[
\left( \| \zeta_{t}^{\nu,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,h} \|_{L^{2}}^{2} \right)^{\frac{C}{2}} \leq \left( e^{-c_{1}M(t)} (\| \zeta_{0}^{\nu,h} \|_{L^{2}}^{2} + \| \zeta_{0}^{w,h} \|_{L^{2}}^{2} ) \right) \\
+ c_{2} \left( \sup_{t \in [0, t]} \| \zeta_{t}^{\nu} \|_{L^{2}} \right) \left( \sup_{t \in [0, t]} (\| \nu \|_{L^{2}} + \| \nu \|_{W^{1,1}}) \int_{0}^{t} e^{-c_{1}M(t-r)} (\| \nu \|_{W^{1,1}} + \| \nu \|_{W^{1,1}}) \, dr \right)^{\frac{C}{2}} \\
\leq c(p, C_{2}, C_{2}, C_{1}, t) \| \zeta_{t}^{\nu,h} \|_{L^{2}} (e^{-c_{1}M(t)} + \frac{1}{M} e^{M(t^{2})} (\| \nu \|_{W^{1,1}} + \| \nu \|_{W^{1,1}})^{2}) \tag{3.75}
\]
by the definition of \(\zeta_{t} = \zeta_{0, t}^{\nu} \), Young’s inequality for convolution, Theorem 2.1 (ii) with \(\lambda = 1\), (3.21), (3.3), and finally taking \(\eta > 0\) smaller if necessary so that \(\Theta(p^{2}/16) \leq \lambda\).

Next, we work on (3.70a), (3.70b) and first estimate
\[
-2 \int (\mathcal{K} \nu \cdot \nu) \zeta_{t}^{\nu,h} \cdot \zeta_{t}^{\nu,l} - 2 \int (\mathcal{K} \zeta_{t}^{\nu} \nu) \nu \cdot \zeta_{t}^{\nu,l} \leq c(\| \nu \|_{W^{1,1}}^{2} + \| \zeta_{t}^{\nu,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{\nu,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{v} \|_{L^{2}}^{2}) \| \zeta_{t}^{\nu,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{\nu,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{v} \|_{L^{2}}^{2}) \tag{3.76}
\]
by Hölder’s inequalities, the embeddings of \(H^{3/2} \hookrightarrow L^{\infty}, H^{1} \hookrightarrow L^{4}\) and \(\mathbb{H}^{1/2} \hookrightarrow L^{4}\) and Poincaré’s, Gagliardo–Nirenberg’s and Young’s inequalities. Next,
\[
-2 \chi \int \Delta_{t} \zeta_{t}^{w,l} \cdot \zeta_{t}^{v,l} \leq \chi(\| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{v,l} \|_{L^{2}}^{2}) \tag{3.77}
\]
by Young’s inequality. Next,
\[
-2 \int (\mathcal{K} \nu \cdot \nu) \zeta_{t}^{w,h} \cdot \zeta_{t}^{w,l} - 2 \int (\mathcal{K} \zeta_{t}^{\nu} \nu) \nu \cdot \zeta_{t}^{w,l} \leq c(\| \nu \|_{W^{1,1}}^{2} + \| \zeta_{t}^{w,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{v} \|_{L^{2}}^{2}) \| \zeta_{t}^{w,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{v} \|_{L^{2}}^{2}) \tag{3.78}
\]
by Hölder’s inequalities, the embeddings of \(H^{3/2} \hookrightarrow L^{\infty}, H^{1} \hookrightarrow L^{4}\) and \(\mathbb{H}^{1/2} \hookrightarrow L^{4}\) and Poincaré’s, Gagliardo–Nirenberg’s and Young’s inequalities. Finally,
\[
-2 \chi \int \zeta_{t}^{w,l} \cdot \zeta_{t}^{w,l} \leq \chi(\| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2}) \tag{3.79}
\]
by Young’s inequality. Applying (3.76)–(3.79) in (3.70a), (3.70b) gives
\[
\partial_{t} (\| \zeta_{t}^{v,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2}) \leq \left( -2\mu - \chi + \epsilon \right) \| \zeta_{t}^{v,l} \|_{L^{2}}^{2} + \left( -2\gamma + \chi + \epsilon \right) \| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \\
+ c(\| \nu \|_{L^{2}}^{2} + \| \nu \|_{W^{1,1}}^{2}) (\| \zeta_{t}^{w,h} \|_{L^{2}}^{2} + \| \zeta_{t}^{w,l} \|_{L^{2}}^{2} + \| \zeta_{t}^{v} \|_{L^{2}}^{2})) \tag{3.80}
\]
by Hölder’s inequalities, the embeddings of \(H^{3/2} \hookrightarrow L^{\infty}, H^{1} \hookrightarrow L^{4}\) and \(\mathbb{H}^{1/2} \hookrightarrow L^{4}\) and Poincaré’s, Gagliardo–Nirenberg’s and Young’s inequalities.
Using the hypothesis that $-2\gamma + \chi < 0$, taking $\epsilon > 0$ small so that $-2\mu - \chi + \epsilon < 0$ and $-2\gamma + \chi + \epsilon < 0$, relying on interpolation inequality, integrating over $[0, t]$ and raising to power of $\nu$, we have

$$
\|\zeta^v_t\|_{L^2}^\nu + \|\zeta^w_t\|_{L^2}^\nu \leq C \left( \int_0^t e^{-\lambda (t-r)^\nu} \left( \int_0^t |\nabla \zeta^v_{t-r} + |\zeta^w_{t-r}| \right)^\nu dr \right)^{\frac{\nu}{\nu-1}} \left( \int_0^t \|\zeta^v_{t-r}\|_{L^2} + \|\zeta^w_{t-r}\|_{L^2} \right)^{\frac{\nu}{\nu-1}}
$$

where we used that $\zeta^v_0 = \zeta^w_0 = 0$ and Hölder’s inequality. Taking expected values and using Hölder’s inequality and (3.75) give

$$
\mathbb{E}[\|\zeta^v_{t}\|_{L^2}^\nu, \|\zeta^w_{t}\|_{L^2}^\nu]^{\nu-1} \leq \mathbb{E}\left[ \left( \int_0^t e^{-\lambda (t-r)^\nu} \left( \int_0^t |\nabla \zeta^v_{t-r} + |\zeta^w_{t-r}| \right)^\nu dr \right)^\nu \right] \mathbb{E}\left[ \left( \int_0^t \|\zeta^v_{t-r}\|_{L^2} + \|\zeta^w_{t-r}\|_{L^2} \right)^\nu \right]^{\nu-1}.
$$

Thus,

$$
\mathbb{E}[\|J^v_{0,T}(\text{Id} - \pi)\phi, J^w_{0,T}(\text{Id} - \pi)\phi\|_{L^2}^\nu]^{\nu-1} \leq \mathbb{E}\left[ \left( \int_0^t e^{-\lambda (t-r)^\nu} \left( \int_0^t |\nabla \zeta^v_{t-r} + |\zeta^w_{t-r}| \right)^\nu dr \right)^\nu \right] \mathbb{E}\left[ \left( \int_0^t \|\zeta^v_{t-r}\|_{L^2} + \|\zeta^w_{t-r}\|_{L^2} \right)^\nu \right]^{\nu-1}
$$

for $M > 0$ sufficiently large, $n$ sufficiently small so that $\beta \eta \leq \lambda$, where we used that $(\text{Id} - \pi)\phi = \zeta^v, \zeta^w = J^v_{0,T}\zeta^v, \zeta^v = \zeta^v + \zeta^v\phi, \zeta^w = \zeta^w + \zeta^w\phi$, (3.80), (3.75), (3.21) and (3.3). This completes the proof of Corollary 2.3. □

References


