GLOBAL MARTINGALE SOLUTION FOR THE STOCHASTIC BOUSSINESQ SYSTEM WITH ZERO DISSIPATION

KAZUO YAMAZAKI

Abstract. We study the two-dimensional stochastic Boussinesq system with zero dissipation and multiplicative noise. We show the existence of a martingale solution by a priori estimates using stochastic calculus, and applications of Prokhorov’s, Skorokhod’s and Martingale Representation theorems. Due to the lack of dissipation, the proof requires higher regularity estimates, taking advantage of the structure of the nonlinear term. Moreover, we obtain the existence of the pressure term via an application of de Rham’s theorem for processes.

Keywords: Navier-Stokes equations, Boussinesq system, martingale solution, Prokhorov’s theorem, Skorokhod’s theorem

1 Introduction

We study the equations of thermohydraulics, precisely in a spatial domain \( D = (0, L) \times (0, 1), \) with \( L > 0 \), filled with fluid,

\[ \begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p &= \nu \Delta u + e_2 (T - T_1) + f_1, \\
\frac{\partial T}{\partial t} + (u \cdot \nabla) T &= \kappa \Delta T + f_2, \\
\nabla \cdot u &= 0,
\end{align*} \]

where we denote by \( u(t, x) = (u_1, u_2)(t, x), p(t, x), T(t, x) \) the velocity vector, pressure and temperature scalar fields respectively, with \( e_i, i = 1, 2 \) the canonical basis of \( \mathbb{R}^2 \), \( \nu, \kappa \geq 0 \) the viscosity and thermal diffusivity related by the Grashof number \( \frac{1}{\nu^2} \), Prandtl number \( \frac{\nu}{\kappa} \) and Rayleigh number \( \frac{1}{\nu \kappa} \). Moreover, we denote by \( f_1, f_2 \) the random forcing terms, to be specified subsequently, and \( T_1 \) the temperature at \( \{ x = (x_1, x_2) \in D : x_2 = 1 \} \) (see [25] Chapter III 3.5).

Hereafter, for brevity we denote \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x_i} \) by \( \partial_t, \partial_i, i = 1, 2 \) and consider the boundary conditions of

\[ \begin{align*}
u(t, x)|_{x_2=0} &= u(t, x)|_{x_2=1} = 0, \\
T(t, x)|_{x_2=0} &\triangleq T_0, \quad T(t, x)|_{x_2=1} = T_1 = T_0 - 1,
\end{align*} \]

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2Department of Mathematics and Statistics, Washington State University, Pullman, WA 99164-3113, U.S.A., Phone: 509-335-9812, E-mail: kyamazaki@math.wsu.edu.
and that $u, T, p$ and the first derivatives of $u, p$ are periodic of period $L$ in the $x_1$-direction. Following [35] Chapter III 3.5, we may subtract from $T$ the pure conduction solution and reconsider instead

$$\theta \triangleq T - T_0 - x_2(T_1 - T_0) = T - T_0 + x_2,$$

$$\pi \triangleq p - (x_2 + \frac{x_2^2}{2})(T_0 - T_1) = p - (x_2 + \frac{x_2^2}{2}),$$

and also split the forcing terms to slowly and rapidly fluctuating parts so that

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi = \nu \Delta u + \theta e_2 + F_1(u, \theta, t) + G_1(u, \xi),$$

$$\partial_t \theta + (u \cdot \nabla)\theta = \kappa \Delta \theta + u_2 + F_2(u, \theta, t) + G_2(\theta, \xi),$$

$$\nabla \cdot u = 0,$$

(3a)

(3b)

(3c)

with an additional boundary condition to (2a) of

$$\theta|_{x_2=0} = \theta|_{x_2=1} = 0,$$

(4)

where $\xi$ is a Gaussian random field, white noise in time, to be specified subsequently.

The deterministic Boussinesq system (3a)-(3c), i.e. with $G_1 \equiv G_2 \equiv 0$, has been studied intensively in both viscous and inviscid cases (e.g. [9]). We note in particular that the question of singularity of $\nabla \theta$ in the partially viscous case $\kappa = 0, \nu > 0$ was raised as an outstanding open problem in [28] and the author in [8] provided an affirmative resolution to this case (see also [4, 6, 12, 17, 18, 20, 21, 22, 36, 38] for further extensive results in this direction of research).

Stochastic partial differential equations of fluid mechanics have been studied by many (e.g. [15] and references found therein). In particular, the author in [14] studied the two-dimensional stochastic Boussinesq system with both dissipation and diffusion, additive noise only on the velocity field equation, and obtained the existence and uniqueness of its solution and invariant measures for the associated semigroup. Moreover, the authors in [37] studied the stochastic Boussinesq system with partial viscosity and additive noise and obtained the global well-posedness type results, following the work of the deterministic case in [8, 21]. More recently, the authors in [5] studied the stochastic Boussinesq system with multiplicative noise but with full dissipation and diffusion; we also refer to [10, 13]; as we will see, none of these results may be applied to the model under our consideration, in particular due to the lack of dissipation (see Theorem 2.1).

In the next section, we set up notations, state our main result and key lemmas needed to for its proof; thereafter, we present the proof.

2. Preliminaries and Statement of Main Result

Let us write $X_1 \approx X_2, X_1 \preceq X_2$ when there exists an insignificant constant $c \geq 0$ such that $X_1 = cX_2, X_1 \preceq cX_2$ respectively whereas if a constant $c$ depends on $a, b$, then we write $X_1 \approx_{a,b} X_2, X_1 \preceq_{a,b} X_2$. We denote the vorticity $\omega \triangleq \nabla \times u$ and the standard Lebesgue and Sobolev spaces by $L^p, W^{m,p}, H^m \triangleq W^{m,2}$; for brevity, we write in bold when vector-valued, e.g. $L^2(D) = (L^2(D))^2$, and also simply $L^2$ when no confusion arises. Following the previous work of [5], we define

$$H_1 \triangleq \{ v \in L^2(D) : \nabla \cdot v = 0, v_2|_{x_2=0} = v_2|_{x_2=1} = 0, v_1|_{x_1=0} = v_1|_{x_1=L} \},$$

$$H_2 \triangleq L^2(D),$$
and $H \triangleq H_1 \times H_2$ where we endow $H_1, H_2$ with an inner product, a norm of
\[(u, v) \triangleq \int_D u \cdot v dx, \quad |v|^2 \triangleq (v, v) \quad \forall u, v \in H_1,\]
respectively and similarly for $H_2$ so that for $\Psi \triangleq (u, \theta), \Phi \triangleq (v, \zeta)$,
\[(\Psi, \Phi) \triangleq (u, v) + (\theta, \zeta), \quad (\Phi, \Phi) \triangleq |\Phi|^2 = |v|^2 + |\zeta|^2.\]
Moreover, we endow a space $V_2 \triangleq \{\zeta \in H^1(D) : \zeta|_{x_2=0} = \zeta|_{x_2=1} = 0, \zeta \text{ is periodic in } x_1 - \text{ direction with period } L\}$, a scalar product, a norm of
\[((\zeta, \eta)) \triangleq \int_D \nabla \zeta \cdot \nabla \eta dx, \quad ((\zeta, \zeta)) \triangleq \|\zeta\| \quad \forall \zeta, \eta \in V_2,\]
respectively and
\[V_1 \triangleq \{v \in (V_2)^2 : \nabla \cdot v = 0, v \cdot n|_{\partial D} = 0\},\]
where $n$ is an outward unit normal vector, also with the inner product and the norm of $((\cdot, \cdot)), \|\cdot\|$ respectively. We let $V \triangleq V_1 \times V_2$ endowed with its norm for $\Psi = (u, \theta), \Phi = (v, \zeta)$,
\[((\Psi, \Phi)) \triangleq ((u, v)) + ((\theta, \zeta)), \quad ((\Phi, \Phi)) \triangleq |\Phi|^2 = |v|^2 + |\zeta|^2.\]
Next, we denote by $D(A_{\nu, \kappa}) \triangleq D(A_{1, \nu}) \times D(A_{2, \kappa})$ where $A_{1, \nu} : D(A_{1, \nu}) \mapsto H_1, A_{2, \kappa} : D(A_{2, \kappa}) \mapsto H_2$ are the Stokes and Laplace operators respectively for which
\[D(A_{1, \nu}) \triangleq \{v \in V_1 \cap (H^2(D))^2 : \partial_1 v|_{x_1=0} = \partial_1 v|_{x_1=L}\},\]
\[\langle A_{1, \nu} v, u \rangle \triangleq \nu((v, u)), \quad \forall u, v \in D(A_{1, \nu}),\]
\[D(A_{2, \kappa}) \triangleq \{\zeta \in V_2 \cap H^2(D) : \partial_1 \zeta|_{x_1=0} = \partial_1 \zeta|_{x_1=L}\},\]
\[\langle A_{2, \kappa} \zeta, \theta \rangle \triangleq \kappa((\zeta, \theta)), \quad \forall \theta, \zeta \in D(A_{2, \kappa}),\]
and
\[\langle A_{\nu, \kappa} \Psi, \Phi \rangle \triangleq \langle A_{1, \nu} u, v \rangle + \langle A_{2, \kappa} \theta, \zeta \rangle \quad \forall \Psi = (u, \theta), \Phi = (v, \zeta).\]
We note that we can define $D(A_{\nu, \kappa})$, $\alpha \in \mathbb{R}$ (e.g. pg. 58 [35]) which is equivalent to $H^\alpha(D)$ with its inner product and a norm denoted by $((\cdot, \cdot))_{D(A_{\nu, \kappa})} = ((\cdot, \cdot))_{D(A_{\nu', \kappa'})}$
so that in particular we have $|\Phi| = |A_{\nu, \kappa}^\alpha \Phi|$. We also recall that the embedding $H^s \hookrightarrow H^{s-\epsilon}$ is compact $\forall \epsilon > 0$ (e.g. Theorem 16.1 [27]). We denote by $\langle \cdot, \cdot \rangle$ the duality pairing such as $\langle V, V' \rangle$ while a quadratic variation by $\langle \langle \cdot, \cdot \rangle \rangle$. Moreover, we write
\[(B_1(v, u), \nu) \triangleq \int_D ((v \cdot \nabla) v) \cdot u dx, \quad (B_2(v, \zeta), \theta) \triangleq \int_D (v \cdot \nabla) \zeta \theta dx,\]
and for $\Psi = (u, \theta), \Phi = (v, \zeta), B(\Phi) = B((v, \zeta)) \triangleq (B_1(v, u), B_2(v, \zeta))^T$,
\[F(\Phi, t) \triangleq \begin{pmatrix} F_1(\Phi, t) \\ F_2(\Phi, t) \end{pmatrix}, \quad G(\Phi, \xi, t) \triangleq \begin{pmatrix} G_1(v, \xi)(t) \\ G_2(\zeta, \xi)(t) \end{pmatrix}, \quad R(\Phi) = R(v, \zeta) \triangleq \begin{pmatrix} \zeta \\ v \end{pmatrix}.\]
We now let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]), P)$ be a stochastic basis with expectation $E, K$ another separable Hilbert space, $W_t(i)$, $i = 1, 2$ cylindrical $K$-valued Wiener processes, defined on the stochastic basis. We denote by $L^p(\Omega, \mathcal{F}, P; L^r(0, T; L^s(D)))$, $p, r, s \in$...
endowed with its norm
\[ \| \phi \|_{L^p(\Omega; F; P; L^r(0, T; L^s(D)))} = \left( E \| \phi(\cdot, \cdot) \|_{L^r(0, T; L^s(D))}^p \right)^{\frac{1}{p}} \]
(see pg. 21 [11]). We denote by \( H \subset \mathcal{H} \).

Theorem 2.1. Suppose \( T > 0, y_0(x) \equiv y(0, x) = (u(0, x), \theta(0, x)) \in H_2(\nu) = 0 \), \( \kappa > 0 \). Moreover, suppose that \( F_i : H \times (0, T) \mapsto H_i, i = 1, 2 \) are nonlinear mappings, continuous in all variables such that

\[ |F_1(y, t)| \lesssim 1 + |u|, \quad |F_2(y, t)| \lesssim 1 + |\theta|, \]  
(5a)

\[ \| \nabla \times F_1(y, t) \| \lesssim 1 + \| u \|_{V_1}, \]  
(5b)

while \( G_i, i = 1, 2 \) are mappings from \( V_i \) to \( L_2(K, Y) \) respectively such that

\[ G_1(u, \xi)(t, x) \equiv \sum_{i=1}^{\infty} c_i^j u(t, x) \partial_i \beta_i^j(t), \quad G_2(\theta, \xi)(t, x) \equiv \sum_{i=1}^{\infty} c_i^j \theta(t, x) \partial_i \beta_i^j(t), \]  
(6)

where \( \{ \beta_i^j \}_{i=1}^{\infty}, j = 1, 2 \) are independent Brownian Motions, \( \{ c_i^j \}_{i=1}^{\infty} \subset L(V_j, H_j), j = 1, 2 \) are linear operators such that

\[ \sum_{i=1}^{\infty} |c_i^j u|^2 \lesssim |u|^2, \quad \sum_{i=1}^{\infty} |c_i^j \theta|^2 \lesssim |\theta|^2, \]  
(7a)

\[ \sum_{i=1}^{\infty} \| \nabla \times (c_i^j u) \|^2 \lesssim |w|^2 + |u|^2, \]  
(7b)

and \( G_i, i = 1, 2 \) extend to Lipschitz continuous mappings \( G_i : H_i \mapsto L_2(K, V'_i) \) such that

\[ |G_i(u, \xi)|^2_{L_2(K, V'_i)} \lesssim 1 + |\xi|^2_{H_i} \]  
(8)

\( \forall t \in [0, T] \). Then there exists a martingale solution to (3a)-(3c) subjected to (2a) and (4), namely a stochastic basis \( (\Omega, F_i, (F_i)_{t \in [0, T]}, P) \), a cylindrical Wiener process \( W = (W_1, W_2) \) on the space \( K \) and a progressively measurable process \( y \equiv (u, \theta) \) such that for \( \alpha > 1 \), with \( P \)-a.e. paths,

\[ y \in C([0, T]; H_w) \cap L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; V) \]

and satisfies \( \forall t \in [0, T] \) and \( \Phi = (v, \zeta) \in D(A_{t, \kappa}^\frac{\alpha}{2}) \),

\[ (y(t), \Phi) + \int_0^t \langle B(y(\tau)) + A_{0, \kappa}(v(\tau), \Phi) \rangle d\tau \]

\[ = (y_0, \Phi) + \int_0^t \langle R(y(\tau)) + F(y(\tau), \Phi) \rangle d\tau + \left( \int_0^t G(y(\tau)) dW, \Phi \right), \]  
(9)

almost surely where we denote

\[ G(y(t)) dW \equiv \left( \sum_{i=1}^{\infty} c_i^j u(t, x) d\beta_i^j \right) \]  

Finally, there exists a unique \( \pi \in L^1(\Omega, F; P; W^{-1, \infty}(0, T; L^2(D))) \) such that \( \int_D \pi dx = 0 \) in \((C_c^\infty([0, T]))'\) and (3a)-(3c) holds in the distributional sense.
Remark 2.1.  

1. In the statement of Theorem 2.1, by \( \Phi = (v, \zeta) \in D(A^\frac{3}{2}_{1, \kappa}) \), we indicate that \( \Phi \in D(A^\frac{2}{1}_{1, \kappa}) \) with \( \nu = 1 \); our choice of \( 1 \) is not crucial as we just need to denote so because we investigate (3a)-(3c) with \( \nu = 0 \). Similarly, by \( A_0, \nu(y(\tau)) \) we indicate \( (0, A_{2, \nu})(\theta)^T \).

2. The condition on the noise may be generalized somewhat (see [3, 10]); we chose to state in the present form for simplicity.

3. Our proof was inspired by the work of [2, 3, 16, 31]. We also mention that it was inspired by the recent surge in the investigation of the two-dimensional deterministic MHD system with zero dissipation and full diffusion (e.g. [7] and references found therein). Due to the complex structure of the non-linear terms in the MHD system, our proof does not seem to go through in the aim to obtain an analogous result for the two-dimensional stochastic MHD system with zero dissipation, or even three-dimensional Boussinesq system with zero dissipation.

4. In the proof, we actually show that \( \|\cdot\|_X \) is compact. Moreover, for \( p > 1, \gamma \in (0, 1) \), we denote by \( W^{\gamma, p}(0, T; X) \) the Sobolev space \( \gamma \in (0, 1) \), we denote by \( W^{\gamma, p}(0, T; X) \) the Sobolev space \( \forall \phi \in L^p(0, T; X) \) endowed with its norm

\[
\|\phi\|_{W^{\gamma, p}(0, T; X)} \triangleq \int_0^T \|\phi(t)\|_X^p dt + \int_0^T \int_0^T \frac{\|\phi(t) - \phi(s)\|_X^p}{|t-s|^{1+\gamma p}} dtds.
\]

The following lemmas are due to [16]:

**Lemma 2.2.** [16] Let \( p \geq 2, \gamma \in (0, \frac{1}{2}) \). If \( f \) is a progressively measurable process and \( f \in L^p(\Omega \times [0, T]; L_2(K, X)) \), then \( I(f) \in L^p(\Omega, W^{\gamma, p}(0, T; X)) \) and

\[
E[\|I(f)\|_{W^{\gamma, p}(0, T; X)}^p] \lesssim_{p, \gamma} E[\int_0^T \|f\|_{L_2(K, X)}^p dt].
\]

**Lemma 2.3.** [16] Let \( B, B_0, B_1 \) be Banach spaces such that \( B_0 \subset B \subset B_1 \) and the embedding \( B_0 \hookrightarrow B \) is compact. Moreover, let \( p \in (1, \infty), \gamma \in (0, 1) \) and \( X = L^p(0, T; B_0) \cap W^{\gamma, p}(0, T; B_1) \). Then the embedding \( X \hookrightarrow L^p(0, T; B) \) is compact.

**Lemma 2.4.** [16] Let \( B_1, B_2 \) be Banach spaces such that the embedding \( B_1 \hookrightarrow B_2 \) is compact, \( \gamma \in (0, 1), p > 1 \) satisfy \( \gamma p > 1 \). Then the embedding \( W^{\gamma, p}(0, T; B_1) \hookrightarrow C([0, T]; B_2) \) is compact.

Let us also state de la Vallée-Poussin theorem for convenience:
Lemma 2.5. (e.g. [27]) The family \( \{X_\alpha\}_{\alpha \in A} \subset L^1(\mu) \) is uniformly integrable if and only if there exists \( f(\tau) \geq 0 \), increasing and convex such that
\[
\lim_{\tau \to \infty} \frac{f(\tau)}{\tau} = \infty \quad \text{and} \quad \sup_{\alpha} E[f(|X_\alpha|)] < \infty.
\]

The following generalization of de Rham’s Theorem to processes is due to [24].

Lemma 2.6. ([24]) Let \( D \subset \mathbb{R}^2 \) be bounded, connected, Lipschitz and open and \((\Omega, \mathcal{F}, P)\) be a complete probability space. Suppose that for \( r_0, r_1 \in [1, \infty], s_1 \in \mathbb{Z}, k \in L^{r_0}(\Omega, \mathcal{F}, P; W^{s_1,r_1}(0,T; \mathbb{H}^{-1}(D))) \) satisfies for any \( v \in (C_c^\infty(D))^2 \) such that \( \nabla \cdot v = 0, P-a.s. \),
\[
(k,v)(C_c^\infty(D))^2 \times (C_c^\infty(D))^2 = 0 \quad \text{in} \quad (C_c^\infty(0,T))^2.
\]
Then there exists a unique \( \pi \in L^{r_0}(\Omega, \mathcal{F}, P; W^{s_1,r_1}(0,T; L^2(D))) \) such that \( P-a.s. \),
\[
\nabla \pi = k \quad \text{in} \quad ((C_c^\infty([0,T] \times D))')^2, \quad I_D \pi dx = 0 \quad \text{in} \quad (C_c^\infty([0,T]))'.
\]

3. Proof of Theorem 2.1

3.1. A priori estimates. We fix \( T > 0 \) and define \( \{c_j\}, \{d_j\}, \{e_j\} \) to satisfy
\[
(v, c_j)_2 = (v, A_{t,\nu}^\alpha c_j) = \lambda_j^{1,\alpha}(v, c_j) \quad \forall v \in D(A_{t,\nu}^\alpha), (1a)\\
(\zeta, d_j)_2 = (\zeta, A_{t,\nu}^{\alpha,\nu} d_j) = \lambda_j^{2,\alpha}(\zeta, d_j) \quad \forall \zeta \in D(A_{t,\nu}^{\alpha,\nu}), (1b)\\
((v, e_j))_2 = (v, A_{t,\nu}^{\alpha,\nu} e_j) = \lambda_j^{2,\nu}(v, e_j) \quad \forall v \in V_1, (1c)
\]
where we may assume that \( \{c_j\}, \{d_j\} \) are orthonormal basis in \( D(A_{t,\nu}^\alpha), D(A_{t,\nu}^{\alpha,\nu}) \) respectively. It follows that the eigenfunctions of \( D(A_{t,\nu}^\alpha) \equiv D(A_{t,\nu}^\alpha) \times D(A_{t,\nu}^{\alpha,\nu}) \) are \( q_j \equiv q_j^1 \equiv q_j^2 \) where \( q_j^1 \equiv (c_j, 0), q_j^2 \equiv (0, d_j) \) and
\[
(\Phi, d_j)_2 = \lambda_j^{2,\nu}(\Phi, d_j) \quad \forall \Phi = (v, \zeta) \in D(A_{t,\nu}^{\alpha,\nu}). (11)
\]
On \( (\Omega, \mathcal{F}, \mathbb{W}) \) with \( \mathbb{W} = (\mathbb{W}_1, \mathbb{W}_2) \) where we denote the expected value with respect to \( \mathbb{F} \) by \( \mathbb{E} \), we look for \( y^m \equiv (u^m, \theta^m) \) such that \( y^m \in \text{span}(q_j^1)_{j=1}^m, \) on \( [0, t_m], t_m > 0 \) and satisfies
\[
\begin{align*}
dy^m + [B(y^m(t)) + A_{0,\nu} y^m(t)] dt &= [Ry^m(t) + F(y^m(t)) dt + G(y^m(t))d\mathbb{W}, (12a)\\
y_0^m &\equiv y^m(0), y_0^m \to y_0 \text{ strongly in } H \quad (m \to \infty), (12b)
\end{align*}
\]
where
\[
G(y^m(t))d\mathbb{W} \equiv \left( \sum_{i=1}^\infty c_i^1 u^m(t, x)d\mathbb{W}_i^\alpha \right) \left( \sum_{i=1}^\infty c_i^2 \theta^m(t, x)d\mathbb{W}_i^\nu \right)
\]
so that
\[
d(y^m, q_j^1) + (B(y^m(t)) + A_{0,\nu} y^m(t), q_j^1) dt = (Ry^m(t) + F(y^m(t), q_j^1) dt + (G(y^m(t)), q_j^1) d\mathbb{W}. (13)
\]
We denote for simplicity \( u^m \equiv \nabla \times u^m \). Now locally on \([0, t_m]\), possibly \( t_m < T \), the solution is known to exist (e.g. Theorem 2, pg. 121 [33]). To extend to \([0, T]\), we perform a priori estimates. We observe the following identities due to (11):
\[
y^m(t) = \sum_{k=1}^m \sum_{j=1}^m \lambda_j^{k,\alpha}(y^m(t), q_j^k) q_j^k, \quad \text{and} \quad |y^m(t)|^2 = \sum_{k=1}^m \sum_{j=1}^m \lambda_j^{k,\alpha}(y^m, q_j^k)^2. \quad (14)
\]
Proposition 3.1. Under the hypothesis of Theorem 2.1, \( \forall p \in [1, \infty) \), the solution \( y^m \) to (13) satisfies

\[
\mathcal{E}[ \sup_{t \in [0,T]} |y^m(t)|^p] + (2\kappa)p\mathcal{E}[\left( \int_0^T \|\theta^m\|^2 d\tau \right)^p] \lesssim_p 1.
\]

Proof. We apply Ito’s formula on (13), multiply by \( \lambda_j^{k,\alpha} \), sum over \( j = 1, \ldots, m, k = 1, 2 \) and rely on (14) to obtain

\[
d|y^m|^2 + 2\kappa\|\theta^m\|^2 d\tau = 2(Ry^m(t) + F(y^m, t), y^m)dt + 2(G(y^m(t)), y^m(t))d\mathbb{W}(t)
\]

We now apply Ito’s formula on (15) with \( p \in [4, \infty) \) to obtain

\[
d|y^m|^p + pk|y^m|^{p-2}\|\theta^m\|^2 d\tau = p|y^m|^{p-2}(Ry^m(t) + F(y^m, t), y^m)dt
\]

\[
+ p\left( \frac{p}{2} - 1 \right)|y^m|^{p-4} (G(y^m(t)), y^m)d\mathbb{W}(t)^2
\]

\[
+ p|y^m|^{p-2} \sum_{k=1}^m \sum_{j=1}^m \lambda_j^{k,\alpha} (G(y^m(t)), q_j^k)d\mathbb{W}(t)^2
\]

\[
+ p|y^m|^{p-2} (G(y^m(t)), y^m)d\mathbb{W}(t).
\]

We introduce the standard stopping time:

\[
\tau_N = \begin{cases} 
\inf \{ \tau > 0 : |y^m(\tau)|^2 \geq N^2 \} & \text{if } \{ \omega \in \Omega : |y^m(\tau)|^2 \geq N^2 \} \neq \emptyset, \\
\infty & \text{otherwise.}
\end{cases}
\]

Applying the fact that \( \sum_{k=1}^m \sum_{j=1}^m \lambda_j^{k,\alpha} \|q_j^k\|^2 \lesssim 1 \) to (16), integrating (16) over \([0, \tau]\), taking absolute values on the right hand side, sup over \( \tau \in [0, t \wedge \tau_N] \), taking squares on both sides and then \( \mathcal{E} \), we deduce

\[
\mathcal{E}[ \sup_{\tau \in [0, t \wedge \tau_N]} |y^m(\tau)|^{2p}] + (pk)^2\mathcal{E}[\left( \int_0^{t \wedge \tau_N} |y^m|^{p-2}\|\theta^m\|^2 d\tau \right)^2]
\]

\[
\lesssim_p |y_0^m|^{2p} + \mathcal{E}\left[ \int_0^{t \wedge \tau_N} |y^m|^{p-2} \|Ry^m(\tau) + F(y^m, \tau), y^m)\|d\tau \right]^2
\]

\[
+ \mathcal{E}\left[ \int_0^{t \wedge \tau_N} |y^m|^{p-2} \sum_{i=1}^\infty \|c_i^j u^m\|^2 + \|c_i^j \theta^m\|^2 \right]^{2}
\]

\[
+ \mathcal{E}\left[ \sup_{\tau \in [0, t \wedge \tau_N]} \left( \int_0^{t \wedge \tau_N} |y^m|^{p-2} (G(y^m(\delta)), y^m)d\mathbb{W}(\delta) \right)^2 \right].
\]
We estimate
\[
\mathbb{E}\left( \int_0^{t \wedge \tau_N} |y^m|^{p-2} [(Ry^m(\tau) + F(y^m, \tau), y^m)] d\tau \right)^2 \]
(18)
\[
\leq \mathbb{E}\left( \int_0^{t \wedge \tau_N} |y^m|^{p-2} [\theta^m c_2 + F_1(u^m, \theta^m, \tau), u^m)] d\tau \right)^2 \\
+ \mathbb{E}\left( \int_0^{t \wedge \tau_N} |y^m|^{p-2} [(u^m_2 + F_2(u^m, \theta^m, \tau), \theta^m)] d\tau \right)^2 \lesssim \mathbb{E}\left( \int_0^{t \wedge \tau_N} 1 + |y^m|^{2p} d\tau \right)
\]
by Hölder’s inequalities and (5a). Next,
\[
\mathbb{E}\left[ \int_0^{t \wedge \tau_N} |y^m|^{p-2} \sum_{i=1}^{\infty} \|c^i_1 u^m\|_{L^2}^2 + \|c^i_2 \theta^m\|_{L^2}^2 d\tau \right]^2 \lesssim \mathbb{E}\left( \int_0^{t \wedge \tau_N} |y^m|^{2p} d\tau \right)
\]
(19)
by (7a) and Hölder’s inequalities. Finally,
\[
\mathbb{E}\left[ \sup_{\tau \in [0, t \wedge \tau_N]} \left( \int_0^{\tau} |y^m|^{p-2} (G(y^m(\delta), y^m) dW(\delta)) \right)^2 \right] \\
\lesssim \sum_{i=1}^{\infty} \mathbb{E}\left( \int_0^{t \wedge \tau_N} |y^m|^{2p-4} |(c^i_1 u^m(x, \delta), u^m)|^2 + |y^m|^{2p-4} |(c^i_2 \theta^m(x, \delta), \theta^m)|^2 d\delta \right) \\
\lesssim_p \mathbb{E}\left( \int_0^{t \wedge \tau_N} |y^m|^{2p} d\tau \right)
\]
(20)
by Burkholder-Davis-Gundy and Hölder’s inequalities and (7a). Applying (18), (19) and (20) in (17) and using Gronwall’s inequality type argument, we obtain
\[
\mathbb{E}\left[ \sup_{\tau \in [0, t \wedge \tau_N]} |y^m(\tau)|^p \right] \lesssim_p 1
\]
(21)
\[
\forall \ p \in [8, \infty). \text{ By Hölder’s inequality, (21) holds for } p \in [1, \infty). \text{ Moreover, because the bound is independent of } m, N, \text{ we may let } N \to \infty \text{ to obtain the existence and the bound on } [0, T].
\]
Now we integrate (15) over [0, \tau], take absolute values on the right hand side, use (7a), take sup over \( \tau \in [0, T] \), raise to the \( p \)-th power and take \( \mathbb{E} \) to obtain
\[
(2\kappa)^p \mathbb{E}\left[ \left( \int_0^T \|\theta^m\|_{V^2}^2 d\delta \right)^p \right] \\
\lesssim |y^m_0|^{2p} + \mathbb{E}\left[ \left( \int_0^T |(Ry^m(\delta) + F(y^m, \delta), y^m)| d\delta \right)^p \right] \\
+ \mathbb{E}\left[ \left( \sup_{\tau \in [0, T]} \left( \int_0^\tau (G(y^m(\delta), y^m(\delta)) dW(\delta)) \right)^p \right) + \mathbb{E}\left( \left( \int_0^T |y^m|_{L^2}^2 d\delta \right)^p \right) \right].
\]
We estimate
\[
\mathbb{E}\left[ \left( \int_0^T \left| (Ry^m(\delta) + F(y^m, \delta), y^m) \right| d\delta \right)^p \right] \leq \mathbb{E}\left[ \left( \int_0^T \left| (\theta^m \epsilon_2 + F_1(u^m, \theta^m, \delta), u^m) \right| d\delta \right)^p \right] + \mathbb{E}\left[ \left( \int_0^T \left| (u_2^m + F_2(u^m, \theta^m, \delta), \theta^m) \right| d\delta \right)^p \right] \leq \mathbb{E}[1 + \sup_{\delta \in [0,T]} \|y^m(\delta)\|_{L^p}^2] \lesssim_p 1
\]
by Hölder’s inequalities, (5a) and (21) with \( t \wedge \tau_N \) replaced by \( T \).

Next, we have
\[
\mathbb{E}\left[ \left( \int_0^T \|G(y^m(\delta))\| d\mathcal{W}(\delta) \right)^p \right] \leq \mathbb{E}\left[ \left( \int_0^T \|G(y^m(\delta))\| d\mathcal{W}(\delta) \right)^p \right] \lesssim \mathbb{E}[\sup_{\delta \in [0,T]} \|y^m(\delta)\|_{L^p}^2 \lesssim_p 1]
\]
by Burkholder-Davis-Gundy and Hölder’s inequalities, (7a) and (21) with \( t \wedge \tau_N \) replaced by \( T \). Finally, we have
\[
\mathbb{E}\left[ \left( \int_0^T \|y^m\|_{L^2}^4 d\delta \right)^p \right] \leq \mathbb{E}[\sup_{\delta \in [0,T]} \|y^m(\delta)\|_{L^2}^2 \lesssim_p 1]
\]
by (21) with \( t \wedge \tau_N \) replaced by \( T \). Thus, applying (23), (24) and (25) in (22) completes the proof. \( \square \)

We now obtain the higher regularity result:

**Proposition 3.2.** Under the hypothesis of Theorem 2.1, \( \forall \ p \in [1, \infty) \), the solution \( y^m \) to (13) satisfies
\[
\mathbb{E}[\sup_{t \in [0,T]} \|u^m(t)\|_{V_1}] \lesssim_p 1
\]

**Proof.** We first fix \( p \in [2, \infty) \). We go back to the first component of (13) with \( k = 1 \) to consider
\[
d(u^m, c_j) + \langle (u^m \cdot \nabla) u^m, c_j \rangle dt = \langle \theta^m \epsilon_2 + F_1(u^m, \theta^m, t), c_j \rangle dt + (G_1(u^m(t)), c_j) d\mathcal{W}_1.
\]
We multiply by \( \lambda^1_j \), use the identity of
\[
\lambda^1_j(f, g) = \langle f, g \rangle = \langle \nabla \times f, \nabla \times g \rangle + \langle f, g \rangle \ \forall \ f, g \in V_1
\]
(e.g. Proposition 2.1 [23]) to obtain
\[
d((u^m, \nabla \times c_j) + (u^m, c_j)) + \langle (u^m \cdot \nabla) u^m, \nabla \times c_j \rangle + \langle (u^m \cdot \nabla) u^m, c_j \rangle dt = \langle \partial_t \theta^m + \nabla \times F_1(u^m, \theta^m, t), \nabla \times c_j \rangle + \langle \theta^m \epsilon_2 + F_1(u^m, \theta^m, t), c_j \rangle dt
\]
\[
+ \langle \nabla \times G_1(u^m(t)), \nabla \times c_j \rangle + \langle G_1(u^m(t)), c_j \rangle d\mathcal{W}_1(t).
\]
(26)
We multiply (26) by $\lambda_j^{1,\alpha}(u^m(t), c_j)$, sum over $j = 1, \ldots, m$ and use the identity of $u^m(t) = \sum_{j=1}^m \lambda_j^{1,\alpha}(u^m(t), c_j) c_j$ similarly to (14) to deduce
\[
d[\|u^m\|_{L^2}^2 + \|u^m\|_{L^2}^2] = [\partial_1\theta^m + \nabla \times F_1(u^m, \theta^m, t), w^m] + (\theta^m e_2 + F_1(u^m, \theta^m, t), u^m)\] 
\[+ [\nabla \times G_1(u^m(t), w^m) + (G_1(u^m(t), w^m)]dW_1(t).
\]

We integrate over $[0, \tau]$, take absolute values and then sup over $\tau \in [0, T]$, raise to the $p$-th power, and take $E$ to obtain
\[
E[\sup_{\tau \in [0, T]} \|u^m(\tau)\|_{L^2}^2 + \|u^m(\tau)\|_{L^2}^2]^p \leq p\|u^m\|_{L^2}^2 + E\left(\int_0^T |\partial_1\theta^m + \nabla \times F_1(u^m, \theta^m, t), w^m] + (\theta^m e_2 + F_1(u^m, \theta^m, t), u^m)\] \]
\[+ E\left(\sup_{\tau \in [0, T]} \int_0^T (\nabla \times G_1(u^m(\delta), w^m) + (G_1(u^m(\delta), w^m)]dW_1(\delta))\] \].

We estimate
\[
E\left(\int_0^T |\partial_1\theta^m + \nabla \times F_1(u^m, \theta^m, t), w^m] + (\theta^m e_2 + F_1(u^m, \theta^m, t), u^m)\] \]
\[\leq E\left(\int_0^T (1 + |\theta^m|_{V_2} + \|u^m\|_{V_1}) \|u^m\|_{L^2} + (1 + |\theta^m|_{L^2} + \|u^m\|_{L^2}) \|u^m\|_{L^2} d\tau\] \]
\[\leq pE\left(\int_0^T \|u^m\|_{L^2}^2 d\tau + \left(\int_0^T \|u^m\|_{V_1}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|u^m\|_{L^2}^2 d\tau\right)^{\frac{1}{2}} \right)^p \]
\[+ E\left(\left(\int_0^T |\theta^m|_{V_2}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^T \|u^m\|_{L^2}^2 d\tau\right)^{\frac{1}{2}} \right)^p \]
\[\leq pE\left[1 + \int_0^T \|u^m\|_{L^2}^2 d\tau\right]
\]
where we used Hölder’s inequalities, (5a), (5b), Proposition 3.1, Young’s inequalities and that $\|u^m\|_{V_1} \approx \|u^m\|_{L^2} + \|u^m\|_{L^2}$. Moreover, we estimate
\[
E\left(\sup_{\tau \in [0, T]} \int_0^T (\nabla \times G_1(u^m(\delta), w^m) + (G_1(u^m(\delta), w^m)]dW_1(\delta))\] \]
\[\leq E\left(\left(\int_0^T (\nabla \times G_1(u^m(\tau), w^m) + (G_1(u^m(\tau), w^m)]dW_1(\tau))\right)^{\frac{1}{2}} \right)^p \]
\[\leq E\left(\int_0^T \|u^m\|_{L^2}^2 + \|u^m\|_{L^2}^2 d\tau\right)^{\frac{1}{2}} \leq pE[1 + \int_0^T \|u^m\|_{L^2}^2 d\tau]\]
by Burkholder-Davis-Gundy and H"older’s inequalities, (7a), (7b) and Proposition 3.1. Thus, applying (28) and (29) in (27) and using Gronwall’s inequality type argument implies the desired result for $p \in [2, \infty)$. By H"older’s inequality, the obtained bound remains valid for $p \in [1, 2)$.

\[ \square \]

**Proposition 3.3.** Under the hypothesis of Theorem 2.1, for the solution $y^m$ to (13), $\{\mathcal{L}(y^m)\}_{m \in \mathbb{N}}$ is tight in $L^2(0, T; H) \cap C([0, T]; D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})), \beta > 1$.

**Proof.** Due to Lemma 2.3, as we already know by Propositions 3.1 and 3.2 that $y^m \in L^2(0, T; V) \mathbb{P}$-a.s., in order to show the tightness in $L^2(0, T; H)$, we only need to show that $y^m \in W^{\gamma, 2}(0, T; D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}}))$ for some $\gamma \in (0, 1)$.

We integrate (13) over $[0, t]$, sum over $k = 1, 2$, take absolute values on both sides, and then sup over $c_j$ such that $\|c_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1$, $d_j$ such that $\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1$ so that by definition of the norms of $D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})^\gamma \times D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})^\gamma$, we have

\[
\|y^m(t)\|_{W^{\gamma, 2}(0, T; D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})^\gamma)} 
\leq \sup_{\|c_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t \nabla \cdot (u^m \otimes u^m) \, d\tau, c_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t \nabla \cdot (u^m \theta^m) \, d\tau, d_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t A_{\frac{\alpha}{2}} \theta^m d\tau, d_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t A_{\frac{\alpha}{2}} \theta^m d\tau, c_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t \theta^m e_2 + F_1(u^m, \theta^m, \tau) d\tau, c_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t u^m_{22} + F_2(u^m, \theta^m, \tau) d\tau, d_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t G_1(u^m(\tau)) d\mathbb{W}_1(\tau), c_j \right) \right|_{W^{\gamma, 2}(0, T)} 
+ \sup_{\|d_j\|_{D(A_{\frac{\alpha}{2}}^{\frac{\beta}{2}})} \leq 1} \left| \left( \int_0^t G_2(\theta^m(\tau)) d\mathbb{W}_2(\tau), d_j \right) \right|_{W^{\gamma, 2}(0, T)} \triangleq \sum_{i=1}^{9} I_i.
\]

Before we estimate each term, we recall the definition of $W^{1,2}(0, T; B)$ that requires a function $f \in L^p(0, T; B)$ and $\partial_t f \in L^p(0, T; B)$ and also that $W^{1,p}(0, T; B) \subset
Next, clearly as $L^1 \subset D(A_{1,\nu}^{\frac{2}{p}})$, we have

$$E[I_4 + I_5] \lesssim \|u_0^m\|_{L^2} + \|\theta_0^m\|_{L^2} \lesssim 1.$$  

(34)

Next,

$$E[I_6^2] \lesssim E[\|\int_0^t \theta^m e_2 + F_1(u^m, \theta^m, \tau) d\tau\|_{W^{1,2}(0,T; D(A_{1,\nu}^{\frac{2}{p}}))}^2]$$

$$\lesssim E[\sup_{t \in [0,T]} 1 + \|u^m(t)\|_{L^2} + \|\theta^m(t)\|_{L^2}^2] \lesssim 1$$

by (5a) and Proposition 3.1. Similarly

$$E[I_7^2] \lesssim E[\int_0^T \|u_2^m + F_2(u^m, \theta^m, \tau)\|_{L^2}^2 dt] \lesssim 1$$

(36)
by (5a) and Proposition 3.1. Finally

\[
E[I_8^2] \lesssim \sum_{i=1}^{\infty} E[\int_0^t c_i u^m(\tau) d\bar{\beta}^i_1(\tau)\|u^{\gamma,2}(0,T;H_1)\|_W^2]
\]

(37)

\[
\lesssim \gamma E[\int_0^T \|u^m\|_{L^2}^2 dt] \lesssim \gamma E[\sup_{t \in [0,T]} \|u^m(t)\|_{L^2}^2] \lesssim 1
\]

where we used that \(H_1 \subset D(A_{1,\alpha}^{2,\nu})\), Lemma 2.2, (7a) and Proposition 3.1. Similarly

\[
E[I_9^2] \lesssim \sum_{i=1}^{\infty} E[\int_0^t c_i^2 \theta^m(\tau) d\bar{\beta}^i_2(\tau)\|u^{\gamma,2}(0,T;H_2)\|_W^2]
\]

(38)

\[
\lesssim \gamma E[\int_0^T \|\theta^m\|_{L^2}^2 dt] \lesssim \gamma E[\sup_{t \in [0,T]} \|\theta^m(t)\|_{L^2}^2] \lesssim 1
\]

by the fact that \(H_2 \subset D(A_{2,\nu}^{2,\kappa})\), Lemma 2.2, (7a) and Proposition 3.1. Therefore, applying the estimates (31), (32), (33), (34), (35), (36), (37), (38) in (30), we have shown that \(y^m \in L^2(0,T;V) \cap W^{\gamma,2}(0,T;D(A_{1,\alpha}^{2,\nu}'))\); hence, \(\{L(y^m)\}_{m \in \mathbb{N}}\) is tight in \(L^2(0,T;H)\).

Next, by Lemma 2.4 with the fixed \(\alpha > 1\), we take \(\beta > \alpha\) so that the bound of \(y^m \in W^{\gamma,p}(0,T;D(A_{1,\alpha}^{2,\nu}'))\) for \(\gamma \in (0,\frac{1}{2})\) and \(p \in (\frac{1}{2},\infty)\) implies \(\{L(y^m)\}_{m \in \mathbb{N}}\) is tight in \(L^2(0,T;H) \cap C([0,T];D(A_{1,\alpha}^{2,\nu}'))\) for \(\beta > 1\).
Towards this goal, similarly to how we derived (30), we see that

\[ \|y^m(t)\|_{W^{\gamma,p}(0,T;D(A_{1,\nu}^2)^c)} \]

\[ \leq \sup_{\|c_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t \nabla \cdot (u^m \otimes u^m) \, dt \right\|_{W^{\gamma,p}(0,T)} \]

\[ + \sup_{\|d_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t \nabla \cdot (u^m \theta^m) \, dt \right\|_{W^{\gamma,p}(0,T)} \]

\[ + \sup_{\|d_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t A_{2,\nu} \theta^m \, dt \right\|_{W^{\gamma,p}(0,T)} \]

\[ + \sup_{\|d_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t \theta^m v_2 + F_1(u^m, \theta^m, \tau) \, dt \right\|_{W^{\gamma,p}(0,T)} \]

\[ + \sup_{\|d_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t u_2^m + F_2(u^m, \theta^m, \tau) \, dt \right\|_{W^{\gamma,p}(0,T)} \]

\[ + \sup_{\|c_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t G_1(u^m(\tau)) \, d\bar{W}_1(\tau), c_j \right\|_{W^{\gamma,p}(0,T)} \]

\[ + \sup_{\|d_j\|_{\overline{D}(A_{1,\nu}^2)} \leq 1} \left\| \int_0^t G_2(\theta^m(\tau)) \, d\bar{W}_2(\tau), d_j \right\|_{W^{\gamma,p}(0,T)} \]

Due to the Sobolev embedding of $W^{1,2}(\mathbb{R}) \hookrightarrow W^{\gamma,p}(\mathbb{R})$ for $p \in (\frac{1}{2}, \infty)$, the estimates for $II_1, II_2, II_3, II_6, II_7$ follow from the corresponding estimates on $I_1, I_2, I_3, I_6, I_7$ in (31), (32), (33), (35) and (36) respectively; e.g.

\[
E[II_1] = E[\left\| \int_0^t \nabla \cdot (u^m \otimes u^m) \, dt \right\|^2_{W^{\gamma,p}(0,T;D(A_{1,\nu}^2)^c)}] \\
\leq\bar{E}[\left\| \int_0^t \nabla \cdot (u^m \otimes u^m) \, dt \right\|^2_{W^{1,2}(0,T;D(A_{1,\nu}^2)^c)}].
\]

Moreover, the estimates for $II_4, II_5$ are clear, similarly to (34). Next,

\[
E[II_8] \lesssim \sum_{i=1}^{\infty} E[\left\| \int_0^t c_i^m(u^m(\tau) \, d\bar{W}_1(\tau) \right\|^p_{W^{\gamma,p}(0,T;H_1)}] \\
\lesssim_{p,\gamma} \int_0^T \left\| u^m \right\|^p_{L^2} \, dt \lesssim_{p,\gamma} E[\sup_{t \in [0,T]} \left\| u^m(t) \right\|^p_{L^2}] \lesssim 1
\]
by the embedding \( H_1 \subset D(A_{1,\kappa}^{\hat{\beta}}) \), Lemma 2.2, (7a) and Proposition 3.1. Similarly,

\[
\mathbb{E}[II^p_{\gamma}] \lesssim \sum_{i=1}^{\infty} \mathbb{E}[\int_0^T c_i^2 \theta^m(\tau)d\beta^2(\tau)]_{W^\gamma,p(0,T;H_2)}^p \lesssim \gamma_p \mathbb{E}\left[ \int_0^T |\theta^m|^p_{L^2} dt \right] \lesssim \gamma_p \mathbb{E}\left[ \sup_{t \in [0,T]} |\theta^m(t)|_{L^2}^p \right] \lesssim 1
\]

by the embedding \( H_2 \subset D(A_{2,\kappa}^{\hat{\beta}}) \), Lemma 2.2, (7a) and Proposition 3.1. By Hölder's inequality, we conclude that \( y^m \in W^\gamma,p(0,T;D(A_{1,\kappa}^{\hat{\beta}})) \) as desired. \( \square \)

3.2. Application of Prokhorov's and Skorokhod's theorems. By Proposition 3.3, \( \{L(y^m)\}_{m \in \mathbb{N}} \) is tight in \( L^2(0,T;H) \cap C([0,T];D(A_{1,\kappa}^{\hat{\beta}})), \beta > 1 \) and hence by Prokhorov's theorem from \( [29] \), it is relatively compact. Thus, there exists a subsequence, still denoted by \( \{L(y^m)\}_{m \in \mathbb{N}} \) such that it converges weakly in \( L^2(0,T;H) \cap C([0,T];D(A_{1,\kappa}^{\hat{\beta}})) \). Next, by Skorokhod's theorem from \( [32] \), there exists a stochastic basis \( (\Omega, \mathcal{F}, \mathcal{F}_t)_{t \in [0,T]}, P \) and on this basis \( L^2(0,T;H) \cap C([0,T];D(A_{1,\kappa}^{\hat{\beta}})) \)-valued random variables \( \hat{y} \triangleq (\hat{u}, \hat{\theta}), \hat{y}^m \triangleq (\hat{u}^m, \hat{\theta}^m) \) such that \( \hat{y}^m \) has same probability law as \( y^m = (u^m, \theta^m) \) on \( L^2(0,T;H) \cap C([0,T];D(A_{1,\kappa}^{\hat{\beta}})) \) and

\[
\hat{y}^m \to \hat{y} \quad (m \to \infty) \quad \text{in} \quad L^2(0,T;H) \cap C([0,T];D(A_{1,\kappa}^{\hat{\beta}})) \quad P - \text{almost surely.} \quad (40)
\]

Since \( y^m \) and \( \hat{y}^m \) have same laws, \( \forall m, p \in [1, \infty) \), we still have

\[
E\left[ \sup_{t \in [0,T]} \|y^m(t)\|_{L^2}^p \right] + E\left[ \left( \int_0^T \|y^m(t)\|_{L^2}^2 dt \right)^{p/2} \right] \lesssim 1 \quad (41)
\]

due to Propositions 3.1 and 3.2 so that \( \forall p \in [1, \infty) \)

\[
y^m \to \hat{y} \quad (m \to \infty) \quad \text{weak}^* \quad \text{in} \quad L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;H)),
\]

weakly in \( L^p(\Omega, \mathcal{F}, P; L^2(0,T;V)) \)

and \( P\text{-a.s.} \quad \hat{y}(\cdot, \omega) \in L^\infty(0,T;H) \cap L^2(0,T;V) \).

3.3. Application of Martingale Representation theorem. This subsection follows Section 8.4 \( [11] \) closely (see also \( [10] \)) and hence we only sketch it. We let \( \forall m \geq 1, P^m \) be the projection onto the \( \text{span}\{q_j\}_{j=1}^m \), and similarly \( P_1^m, P_2^m \) onto the \( \text{span}\{c_j\}_{j=1}^m, \text{span}\{d_j\}_{j=1}^m \) respectively and define

\[
\hat{M}_m(t) \triangleq \hat{y}^m(t) - P^m \hat{y}^m(0) + \int_0^t P^m[B(\hat{y}^m(\tau)) + A_0,\kappa \hat{y}^m(\tau)] - P^m[R\hat{y}^m(\tau) + F(\hat{y}^m, \tau)]d\tau.
\]

We claim that \( \hat{M}_m \) with trajectories in \( C([0,T];H) \) is a square integrable martingale with respect to \( \sigma\{\hat{y}^m(\tau), \tau \leq t\} \) with a quadratic variation of

\[
\langle \langle \hat{M}_m \rangle \rangle_t = \int_0^t P^m G(\hat{y}^m(\tau)) G(\hat{y}^m(\tau))^* P^m d\tau. \quad (43)
\]
Indeed, we may define
\[ M_m(t) \triangleq y^m(t) - P^m y^m(0) \]
+ \( \int_0^t P^m[B(y^m(\tau)) + A_{0,m} y^m(\tau)] - P^m[R y^m(\tau) + F(y^m, \tau)]d\tau \)
so that integrating (12a) over \([0, t]\) and applying \( P^m \), we see that
\[ M_m(t) = \int_0^t P^m G(y^m(\tau))d\overline{W}(\tau), \]
and hence clearly \( M_m \) with trajectories in \( C([0, T]; H) \) is a square integrable martingale with respect to \( \sigma\{y^m(\tau), \tau \leq t\} \) with quadratic variations
\[ \langle \langle M_m \rangle \rangle_t = \int_0^t P^m G(y^m(\tau))G(y^m(\tau))^* P^m d\tau. \]
Moreover, \( \forall \, s \leq t, t \in [0, T], \) we may let \( \phi \triangleq (\phi_1, \phi_2) \) be a continuous real-valued bounded mapping on \( L^2(0, s; H) \) or \( C([0, s]; D(A^{\frac{\beta}{2}}, s)) \), \( v = (v_1, v_2), \) \( z = (z_1, z_2) \in H^\beta(D), \beta > 1 \) and show that
\[ E[\langle M_m(t) - M_m(s), v \rangle \phi(y^m|_{[0, s]}))] \]
This implies that by definition of \( M_m \), the fact that \( \mathcal{L}(y^m) = \mathcal{L}({\hat{y}}^m) \) and (42)
\[ E[(\tilde{M}_m(t) - \tilde{M}_m(s), v) \phi({\hat{y}}^m|_{[0, s]}))] = 0; \] (44)
therefore, \( \tilde{M}_m(\cdot) \) is a martingale on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)\).
Moreover, since \( \mathcal{L}({\hat{y}}^m) = \mathcal{L}(y^m) \) and hence \( \mathcal{L}(\tilde{M}_m) = \mathcal{L}(M_m) \), we have
\[ E[\|\tilde{M}_m\|_{L^2}^2] \lesssim \mathcal{E}[ \sup_{t \in [0, T]} \|u^m(t)\|_{L^2}^2 + \|\theta^m(t)\|_{L^2}^2] \lesssim 1 \]
by (7a) and Proposition 3.1. Thus, \( \tilde{M}_m \) is also square integrable.
Similarly, using \( \mathcal{L}(y^m) = \mathcal{L}({\hat{y}}^m) \), we can show that
\[ E[\langle (\tilde{M}_m(t), v)(\tilde{M}_m(t), z) - (\hat{M}_m(s), v)(\hat{M}_m(s), z) \]
\[ - \int_s^t (G({\hat{y}}^m(\tau))^* P^m v, G({\hat{y}}^m(\tau))^* P^m z)d\tau \phi({\hat{y}}^m|_{[0, s]}))] = 0. \] (45)
This implies by definition of \( \tilde{M}_m \), component-wise
\[ E[\langle (\tilde{u}^m(t) - P_1^m \tilde{u}^m(0) + \int_0^t P_1^m [(\tilde{u}^m \cdot \nabla) \tilde{u}^m - \tilde{\theta}^m e_2 - F_1(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, v_1 \)
\[ \times (\tilde{u}^m(t) - P_1^m \tilde{u}^m(0) + \int_0^t P_1^m [(\tilde{u}^m \cdot \nabla) \tilde{u}^m - \tilde{\theta}^m e_2 - F_1(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, z_1) \]
\[ - (\tilde{u}^m(s) - P_1^m \tilde{u}^m(0) + \int_0^s P_1^m [(\tilde{u}^m \cdot \nabla) \tilde{u}^m - \tilde{\theta}^m e_2 - F_1(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, v_1) \]
\[ \times (\tilde{u}^m(s) - P_1^m \tilde{u}^m(0) + \int_0^s P_1^m [(\tilde{u}^m \cdot \nabla) \tilde{u}^m - \tilde{\theta}^m e_2 - F_1(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, z_1) \]
\[ - \int_s^t (G_1(\tilde{u}^m, \xi)^* P_1^m v_1, G_1(\tilde{u}^m, \xi)^* P_1^m z_1)d\tau \phi_1({\hat{y}}^m|_{[0, s]}))] = 0. \] (46)
3.3.1. Uniform integrability and point-wise convergence. We justify taking the limit on (46) and (47) and hence (45), similarly to the work in [5]. We let

$$E[(\tilde{\theta}^m(t) - P_2^m\tilde{\theta}^m(0) + \int_0^t P_2^m[A_{2,\kappa}\tilde{\theta}^m + (\tilde{u}^m \cdot \nabla)\tilde{\theta}^m - \tilde{u}^m_2 - F_2(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, v_2) \times (\tilde{\theta}^m(t) - P_2^m\tilde{\theta}^m(0) + \int_0^t P_2^m[A_{2,\kappa}\tilde{\theta}^m + (\tilde{u}^m \cdot \nabla)\tilde{\theta}^m - \tilde{u}^m_2 - F_2(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, z_2) - (\tilde{\theta}^m(s) - P_2^m\tilde{\theta}^m(0) + \int_0^s P_2^m[A_{2,\kappa}\tilde{\theta}^m + (\tilde{u}^m \cdot \nabla)\tilde{\theta}^m - \tilde{u}^m_2 - F_2(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, v_2) \times (\tilde{\theta}^m(s) - P_2^m\tilde{\theta}^m(0) + \int_0^s P_2^m[A_{2,\kappa}\tilde{\theta}^m + (\tilde{u}^m \cdot \nabla)\tilde{\theta}^m - \tilde{u}^m_2 - F_2(\tilde{u}^m, \tilde{\theta}^m, \tau)]d\tau, z_2) - \int_s^t (G_2(\tilde{\theta}^m, \xi)\ast P_2^m v_2, G_2(\tilde{\theta}^m, \xi)\ast P_2^m z_2)d\tau)\phi_2(\tilde{y}^m_{[0,s]}) = 0. \tag{47}$$

Firstly, for any r > 1, P-a.s.,

$$|g_{a,m}(\omega)| \lesssim \int_0^t \|\tilde{u}^m\|^2_{L^2} \|v_1\|_{H^1} d\tau \int_0^t \|\tilde{u}^m\|^2_{L^2} \|z_1\|_{H^1} d\tau \int_0^t (\int_0^t \|\tilde{u}^m\|^2_{L^2} \|z_1\|_{H^1} d\tau)^r \tag{48}$$

$$\lesssim \int_0^t \|\tilde{u}^m\|^2_{L^2} \|\tilde{u}^m\|_{H^1} d\tau \lesssim \sup_{t \in [0,T]} \|\tilde{u}^m(t)\|_{L^2}^{4r} + \left(\int_0^t \|\tilde{u}^m\|_{H^1} d\tau\right)^{4r}$$

where we used Hölder’s, Gagliardo-Nirenberg and Young’s inequalities. Therefore, $E[|g_{a,m}(\omega)|^r] \lesssim 1$ by (41). This implies $\{g_{a,m}\}_{m \in \mathbb{N}}$ is uniformly integrable due to de la Vallée-Poussin theorem. For point-wise convergence, we may fix $\omega \in \Omega$ such that

$$\tilde{y}^m \to \tilde{y} (m \to \infty) \text{ strongly in } L^2(0, T; H) \cap C([0, T]; D(A_{1,\kappa}^\beta)) \tag{49}$$

due to (40). Thus, by continuity of $\phi_1$, we obtain

$$\phi_1(\tilde{y}^m_{[0,s]}(\omega)) \to \phi_1(\tilde{y}_{[0,s]}(\omega)) \quad (m \to \infty).$$
Hence, $P$-a.s.

$$\int_0^T \int_D [P_1^m \nabla \cdot (\hat{u}^m \otimes \hat{u}^m) - \nabla \cdot (\hat{u} \otimes \hat{u})]v_1 \, dx \, dt$$

$$= - \sum_{i,j=1}^2 \int_0^T \int_D \hat{u}_i^m \hat{u}_{ij}^m (P_1^m - Id) \partial_i v_{1,j} + [(\hat{u}_i^m - \hat{u}_i) \hat{u}_{ij}^m + \hat{u}_i (\hat{u}_{ij}^m - \hat{u}_{ij})] \partial_i v_{1,j} \, dx \, dt$$

$$\lesssim \int_0^T \|\hat{u}_m\|_{L^2} \|\hat{u}^m\|_{H^{1/2}} \|(P_1^m - Id) \nabla v_1\|_{L^2}$$

$$+ \|\hat{u}_m - \hat{u}\|_{L^2} \|\hat{u}^m - \hat{u}\|_{H^{1/2}} \left(\|\hat{u}_m\|_{L^2}^2 + \|\hat{u}\|_{H^{1/2}}^2\right) dt$$

$$\lesssim \sup_{t \in [0,T]} \|\hat{u}_m(t)\|_{L^2} \left(\int_0^T \|\hat{u}_m\|_{H^{1/2}}^2 \, dt\right)^{1/2} \|(P_1^m - Id) \nabla v_1\|_{L^2}$$

$$+ \sup_{t \in [0,T]} \|\hat{u}(t)\|_{L^2} \left(\int_0^T \|\hat{u}_m - \hat{u}\|_{H^{1/2}}^2 \, dt\right)^{1/2} \left(\int_0^T \|\hat{u}_m\|_{H^{1/2}} \, dt\right)^{1/2}$$

$$\lesssim \|(P_1^m - Id) \nabla v_1\|_{L^2} + \|\hat{u}_m - \hat{u}\|_{L^2}^2 (0,T;L^2(D)) \to 0 \quad (m \to \infty)$$

by Hölder’s and Gagliardo-Nirenberg inequalities and (41). Thus, we have shown $g_{u,m}(\omega) \to g_u(\omega)$.

Similarly, we may define

$$g_{\theta,m}(\omega) \triangleq \int_0^t \int_0^{\tau} (P_2^m \nabla \cdot (\hat{u}^m \hat{\theta}^m), v_2) \, d\tau \int_0^{\tau} (P_2^m \nabla \cdot (\hat{u}^m \hat{\theta}^m), z_2) \, d\tau \phi_2(\hat{y}^m|_{[0,s]}),$$

$$g_{\theta}(\omega) \triangleq \int_0^t \int_0^{\tau} (\nabla \cdot (\hat{u} \hat{\theta}), v_2) \, d\tau \int_0^{\tau} (\nabla \cdot (\hat{u} \hat{\theta}), z_2) \, d\tau \phi_2(\hat{y}|_{[0,s]})$$

and obtain similar uniform integrability and point-wise convergence results with same $\omega$ such that (49) holds as in (48) and (50) and hence conclude that $g_{\theta,m}(\omega) \to g_{\theta}(\omega) \quad (m \to \infty)$.

Next, for any $r > 1$, $P$-a.s.,

$$\left| \int_0^t \int_0^\tau (P_2^m A_{2,\kappa} \hat{\theta}^m(\tau), v_2) \, d\tau \int_0^\tau (P_2^m A_{2,\kappa} \hat{\theta}^m(\tau), z_2) \, d\tau \right|^r$$

$$\lesssim \int_0^t \|\hat{\theta}^m\|_{L^2} \|P_2^m v_2\|_{H^1} \, d\tau \int_0^t \|\hat{\theta}^m\|_{L^2} \|P_2^m z_2\|_{H^1} \, d\tau \lesssim 1$$

by Hölder’s inequalities and (41). For the point-wise convergence of the diffusive term, we see that

$$\int_0^t \int_D [P_2^m A_{2,\kappa} \hat{\theta}^m(\tau) - A_{2,\kappa} \hat{\theta}(\tau)] v_2 \, dx \, d\tau$$

$$= \int_0^t \int_D A_{2,\kappa} \hat{\theta}^m(\tau)(P_2^m - Id) v_2 + (A_{2,\kappa} \hat{\theta}^m(\tau) - A_{2,\kappa} \hat{\theta}(\tau)) v_2 \, dx \, d\tau$$
where for the first term, P-a.s.,
\[
\int_0^t \int_D A_{2,\rho} \hat{\theta}^m(\tau) (P_2^m - Id) v_2 dx d\tau \lesssim \| (P_2^m - Id) v_2 \|_{H^1(D)} \to 0 \quad (m \to \infty)
\]
by (41) while because \( \hat{\theta}^m \to \hat{\theta} \) (\( m \to \infty \)) weakly in \( L^p(\Omega, \mathcal{F}, P; L^2(0, T; V_2)) \), as \( A_{2,\rho} v_2 \in H^{\beta-2}(D) \subset H^{-1}(D) \) because \( \beta > 1 \), we also see that P-a.s. the second term in (52) vanishes as \( m \to \infty \).

Next, by (40) and because for any \( r > 1 \)
\[
E[\| \hat{\mu}^m - \hat{\mu} \|_{L^r(0, T; H^4)}^2] + E[\| \hat{\theta}^m - \hat{\theta} \|_{L^r(0, T; H^2)}^2] \lesssim 1,
\]
by de la Vallee-Poussin Theorem. \( \{\| \hat{\mu}^m - \hat{\mu} \|_{L^2(0, T; H^4)}^2\}, \{\| \hat{\theta}^m - \hat{\theta} \|_{L^2(0, T; H^2)}^2\} \) are both uniformly integrable. By Vitali’s convergence theorem (see e.g. [30]), this implies the strong convergence of \( \hat{y}^m \) to \( \hat{y} \) in \( L^2(\Omega, \mathcal{F}, P; L^2(0, T; H)) \) as \( m \to \infty \). Therefore, relabelling subsequence we obtain
\[
\hat{y}^m \to \hat{y} \quad \text{in } H \text{ for a.e. } (\omega, t) \quad (m \to \infty)
\]
with respect to \( dP \times dt \).

Similarly, for any \( r > 1, i = 1, 2 \)
\[
E[\| F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) \|_{L^r(0, T; L^2(D))}^2] \lesssim E[1 + \sup_{t \in [0, T]} \| \hat{\mu}^m(t) \|_{L^2}^2 + \| \hat{\theta}^m(t) \|_{L^2}^2] \lesssim 1
\]
by (5a) and Propositions 3.1. Therefore, \( \{\| F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) \|_{L^2(0, T; L^2(D))}^2\} \) is uniformly integrable by de la Vallee-Poussin Theorem. Moreover,
\[
\| F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) - F_1(\hat{\mu}, \hat{\theta}, \tau) \|_{L^2(0, T; L^2(D))} \to 0 \quad (m \to \infty)
\]
by continuity of \( F_1 : H \times (0, T) \to H \). Thus, by Vitali’s Convergence Theorem,
\[
F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) \to F_1(\hat{\mu}, \hat{\theta}, \tau) \text{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0, T; L^2(D))).
\]
Therefore,
\[
E[\int_0^t \int_D [P_1^m F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) - F_1(\hat{\mu}, \hat{\theta}, \tau)] v_1 dx d\tau] \quad (m \to \infty)
\]
\[
\lesssim \| F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) \|_{L^2(0, T; L^2(D))} \| (P_1^m - Id) v_1 \|_{L^2} + E[\| F_1(\hat{\mu}^m, \hat{\theta}^m, \tau) - F_1(\hat{\mu}, \hat{\theta}, \tau) \|_{L^2(0, T; L^2(D))} \| v_1 \|_{L^2} \to 0 \quad (m \to \infty)
\]
by H"{o}lder’s inequalities. Similarly
\[
E[\int_0^t \int_D [P_2^m F_2(\hat{\mu}^m, \hat{\theta}^m, \tau) - F_2(\hat{\mu}, \hat{\theta}, \tau)] v_2 dx d\tau] \quad (m \to \infty).
\]
Finally, we show that
\[
E\left[ \int_s^t (G_1(\hat{\mu}^m, \xi)^* P_1^m v_1, G_1(\hat{\mu}^m, \xi)^* P_1^m z_1) d\tau \right] \phi_1(\hat{y}^m|_{[0, s]}) \quad (m \to \infty)
\]
\[
- E\left[ \int_s^t (G_1(\hat{\mu}, \xi)^* v_1, G_1(\hat{\mu}, \xi)^* z_1) d\tau \right] \phi_1(\hat{y}|_{[0, s]}) \quad (m \to \infty).
\]
We first write
\[
f_{u, m}(\omega) \triangleq \left( \int_s^t (G_1(\hat{\mu}^m, \xi)^* P_1^m v_1, G_1(\hat{\mu}^m, \xi)^* P_1^m z_1) d\tau \right) \phi_1(\hat{y}^m|_{[0, s]}),
\]
\[
f_1(\omega) \triangleq \left( \int_s^t (G_1(\hat{\mu}, \xi)^* v_1, G_1(\hat{\mu}, \xi)^* z_1) d\tau \right) \phi_1(\hat{y}|_{[0, s]}).
\]
For the fixed $r > 1$ we can compute by (8)
\[
E[|f_{u,m}|^r] \lesssim E\left( \int_0^t 1 + \|\hat{u}^m\|_{H_1}^2 d\tau \right)^r \lesssim E\left[ 1 + \sup_{t \in [0,T]} \|\hat{u}^m(t)\|_{H_1}^r \right] \lesssim 1.
\]
For the point-wise convergence, for the fixed $\omega \in \Omega$ such that (49) holds,
\[
\phi_1(\hat{y}^m|_{[0,s]}(\omega)) \to \phi_1(\hat{y}|_{[0,s]}(\omega)) \quad (m \to \infty)
\]
by continuity of $\phi_1$, so that writing
\[
E[f_{u,m}(\omega)] - E[f_u(\omega)] \quad (55)
\]
we see that the first term vanishes as $m \to \infty$; hence, we only need to show
\[
\int_s^t (G_1(\hat{u}^m, \xi)^* P_{1,s} v_1, G_1(\hat{u}^m, \xi)^* P_{1,z} 1) d\tau - \int_s^t (G_1(\hat{u}, \xi)^* v_1, G_1(\hat{u}, \xi)^* z_1) d\tau \to 0
\]
as $m \to \infty$. For this purpose, we compute $P$-a.s.,
\[
\int_s^t (G_1(\hat{u}^m, \xi)^* P_{1,s} v_1, G_1(\hat{u}^m, \xi)^* P_{1,z} 1) d\tau - \int_s^t (G_1(\hat{u}, \xi)^* v_1, G_1(\hat{u}, \xi)^* z_1) d\tau \leq \|G_1(\hat{u}^m, \xi)^* P_{1,s} v_1 - G_1(\hat{u}, \xi)^* v_1\|_{L^2(s,t;K)} \cdot \|G_1(\hat{u}^m, \xi)^* P_{1,z} 1 - G_1(\hat{u}, \xi)^* z_1\|_{L^2(s,t;K)}
\]
of which
\[
\|G_1(\hat{u}^m, \xi)^* P_{1,s} v_1 - G_1(\hat{u}, \xi)^* v_1\|_{L^2(s,t;K)}^2 \leq \int_s^t \|G_1(\hat{u}^m, \xi)^* (P_{1,s} - Id) v_1\|_{K} + \|G_1(\hat{u}^m, \xi)^* v_1 - G_1(\hat{u}, \xi)^* v_1\|_{K} \|P_{1,s}^m\|_{L^2(s,t;K)} \to 0
\]
as $m \to \infty$ $P$-a.s. where we used (8), hypothesis that $G_1$ may be extended to a Lipschitz mapping from $H_1$ to $L^2(K, V_1')$ and that $\hat{u}^m \to \hat{u}$ ($m \to \infty$) strongly in $L^2(0,T; H_1)$. An identical procedure shows that $P$-a.s.,
\[
\|G_1(\hat{u}^m, \xi)^* P_{1,z} 1 - G_1(\hat{u}, \xi)^* z_1\|_{L^2(s,t;K)} \to 0 \quad (m \to \infty).
\]
Finally, a similar procedure shows that
\[
E\left[ \int_s^t (G_2(\hat{y}^m, \xi)^* P_{2,s} v_2, G_2(\hat{y}^m, \xi)^* P_{2,z} 2) d\tau \right] \phi_2(\hat{y}^m|_{[0,s]}(\omega))
\]
\[
\to E\left[ \int_s^t (G_2(\hat{y}, \xi)^* v_2, G_2(\hat{y}, \xi)^* z_2) d\tau \right] \phi_2(\hat{y}|_{[0,s]}(\omega)) \quad (m \to \infty).
\]
3.3.2. **Continuation of the application of Martingale Representation theorem.** We now take the limit \( \hat{u}^m \to \hat{u}, \hat{\theta}^m \to \hat{\theta}, P_1^m, P_2^m \to Id \quad (m \to \infty) \) on (45) to obtain

\[
E[(\hat{M}(t,v)(\hat{M}(t,z) - (\hat{M}(s,v)(\hat{M}(s,z)) \\
- \int_s^t (G(\hat{y}(\tau))^* v, G(\hat{y}(\tau))^* z) d\tau) \phi(\hat{y}|[0,s])] = 0
\]

where

\[
\hat{M}(t) \triangleq \hat{y}(t) - \hat{y}(0) + \int_0^t B(\hat{y}(\tau)) + A_{0,\kappa}\hat{y}(\tau) - R\hat{y} - F(\hat{y},\tau)d\tau.
\]

Similarly, taking limit \( m \to \infty \) on (44) gives

\[
E[(\hat{M}(t) - \hat{M}(s)v)\phi(\hat{y}|[0,s])] = 0.
\]

This implies \( \hat{M}(t) \) is \( H^{-\beta}(D) \)-valued martingales with respect to \( \sigma\{\hat{y}(\tau), \tau \leq T\} \) having quadratic variations of

\[
\langle \langle \hat{M} \rangle \rangle_t = \int_0^t G(\hat{y}(\tau))G(\hat{y}(\tau))^* d\tau.
\]

Moreover, as we noted already, using \( \mathcal{L}(M_m) = \mathcal{L}(\hat{M}_m) \), we can show \( \forall \ p \in [2, \infty), \)

\[
\sup_{m \in \mathbb{N}} E[\|M_m\|_{L^2(D)}^p] = \sup_{m \in \mathbb{N}} \|M_m\|_{L^2(D)}^p
\]

\[
\lesssim \sum_{i=1}^{\infty} \sup_{m \in \mathbb{N}} \left( \int_{D} \left( \mathbb{E}\left[ \int_0^t P_1^m c_1^m u^m(\tau,x)d\beta^1_1(p)\right]^2 \right)^{\frac{p}{2}} \right)^{\frac{2}{p}}
\]

\[
\quad + \left( \int_{D} \left( \mathbb{E}\left[ \int_0^t P_2^m c_2^m \theta^m(\tau,x)d\beta^2_1(p)\right]^2 \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \lesssim 1
\]

by Minkowski’s inequality for integrals, (4.47) from pg. 114 II, (7a) and Proposition 3.1. Thus, \( \{M_m\}_{m \in \mathbb{N}} \) is uniformly integrable and

\[
\lim_{m \to \infty} E[\|M_m(t)\|_{L^2(D)}^2] = E[\|\hat{M}(t)\|_{L^2(D)}^2];
\]

therefore, \( \hat{M}(t) \) is also a square integrable process.

We now define

\[
\hat{N}_m(t) \triangleq A^{-\frac{2}{p}}_{1,\kappa}\hat{M}_m(t).
\]

As we proved for \( \hat{M}_m \), we can show that \( \hat{N}_m \) is a square integrable martingale on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\), with a quadratic variation of

\[
\langle \langle \hat{N}_m \rangle \rangle_t = \int_0^t A^{-\frac{2}{p}}_{1,\kappa} P^m G(\hat{y}^m(\tau))G(\hat{y}^m(\tau))^* P^m A^{-\frac{2}{p}}_{1,\kappa} d\tau,
\]

and we can show similarly as before that

\[
\hat{N}(t) \triangleq A^{-\frac{2}{p}}_{1,\kappa}\hat{y}(t) - A^{-\frac{2}{p}}_{1,\kappa}\hat{y}(0) + \int_0^t A^{-\frac{2}{p}}_{1,\kappa}[B(\hat{y}(\tau)) + A_{0,\kappa}\hat{y}(\tau) - A^{-\frac{2}{p}}_{1,\kappa}[R\hat{y} + F(\hat{y},\tau)]d\tau
\]

is a square integrable martingale on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\), with a quadratic variation of

\[
\langle \langle \hat{N} \rangle \rangle_t = \int_0^t A^{-\frac{2}{p}}_{1,\kappa} P^m G(\hat{y}^m(\tau))G(\hat{y}^m(\tau))^* P^m A^{-\frac{2}{p}}_{1,\kappa} d\tau.
\]
is a square integrable martingales with respect to $\sigma\{\hat{y}(\tau), \tau \leq t\}$ and 

$$
\langle \langle N \rangle \rangle_t = \int_0^t A_{1,\kappa}^{-\frac{a}{2}} G(\hat{y}(\tau)) G(\hat{y}(\tau))^* A_{1,\kappa}^{-\frac{a}{2}} d\tau.
$$

By the Martingale Representation theorem ([11]), there exists $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{F}_t)_{t \in [0,T]}, \tilde{P})$, a Wiener process $\tilde{W} = (\tilde{W}_1, \tilde{W}_2)$ and a predictable process $\hat{y} \triangleq (\hat{u}, \hat{\theta})$ such that 

$$
A_{1,\kappa}^{-\frac{a}{2}} \hat{y}(t) - A_{1,\kappa}^{-\frac{a}{2}} \hat{y}(0) + \int_0^t A_{1,\kappa}^{-\frac{a}{2}} [B(\hat{y}(\tau)) + A_{0,\kappa} \hat{y}(\tau)]
$$

$$
- A_{1,\kappa}^{-\frac{a}{2}} [R(\hat{y}(\tau) + F(\hat{y}(\tau))] d\tau = A_{1,\kappa}^{-\frac{a}{2}} \int_0^t G(\hat{y}(\tau)) d\tilde{W}.
$$

Taking $A_{1,\kappa}^{-\frac{a}{2}}$ completes the proof of existence of the martingale solution.

### 3.4. Existence of pressure term.

We define 

$$
k \triangleq \partial_t u + \nabla \cdot (u \otimes u) - \theta e_2 - F_1(u, \theta, t) - G_1(u, \xi)(t).
$$

By Lemma 2.6, it suffices to show $k \in L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; H^{-1}(D)))$. We compute 

$$
\|\partial_t u\|_{L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; H^{-1}(D)))} \lesssim E[\|u\|_{L^\infty(0, T; L^2(D))}] \lesssim 1.
$$

Next, 

$$
\|\nabla \cdot (u \otimes u)\|_{L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; H^{-1}(D)))} 
\lesssim E[\int_0^T \|u\|^2_{L^2} d\tau] \lesssim \left( E[\sup_{t \in [0,T]} \|u(t)\|_{L^2}^2] \right)^{\frac{1}{2}} \left( E[\left( \int_0^T \|u\|_{H^1} d\tau \right)^2] \right)^{\frac{1}{2}} \lesssim 1
$$

by Sobolev embedding of $L^1([0, T]) \hookrightarrow W^{-1,\infty}([0, T])$ (cf. pg. 97 [1]), Gagliardo-Nirenberg and Hölder’s inequalities. Next, 

$$
\|\theta e_2\|_{L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; H^{-1}(D)))} \lesssim \|\theta\|_{L^1(\Omega, \mathcal{F}, P, L^1(0,T; L^2(D)))} \lesssim 1
$$

by Sobolev embedding of $L^1([0, T]) \hookrightarrow W^{-1,\infty}((0, T])$. Next, 

$$
\|F_1(u, \theta, t)\|_{L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; H^{-1}(D)))} \lesssim E[\int_0^T \|F_1(u, \theta, t)\|_{L^2}^2 d\tau] \lesssim 1
$$

by Sobolev embedding of $L^1([0, T]) \hookrightarrow W^{-1,\infty}((0, T])$ and (5a). Finally, due to Minkowski’s inequality for integrals, Burkholder-Davis-Gundy inequality and (7a), 

$$
\|G_1(u, \xi)\|_{L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0, T; H^{-1}(D)))} 
\lesssim \left( \sum_{i=1}^\infty \int_0^T c_i^1 u(\tau, x) d\beta_i^1(\tau) \right)_{L^1(\Omega, \mathcal{F}, P, L^\infty(0, T; H^{-1}(D)))}
$$

$$
\lesssim \left( \sum_{i=1}^\infty \int_0^T c_i^1 u(\tau, x) d\beta_i^1(\tau) \right)_{L^2(\Omega, \mathcal{F}, P; L^2(D; L^\infty([0,T])))}
$$

$$
\lesssim \left( \int_D \sum_{i=1}^\infty E[\langle \int_0^T c_i^1 u(\tau, x) d\beta_i^1(\tau) \rangle] dx \right)^{\frac{1}{2}} \lesssim \left( E[\sup_{t \in [0,T]} 1 + \|u(t)\|^2_{L^2}] \right)^{\frac{1}{2}} \lesssim 1.
$$
References


