

A FINITENESS PROPERTY FOR PREPERIODIC POINTS OF CHEBYSHEV POLYNOMIALS

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ABSTRACT. Let K be a number field with algebraic closure \overline{K} , let S be a finite set of places of K containing the archimedean places, and let φ be a Chebyshev polynomial. We prove that if $\alpha \in \overline{K}$ is not preperiodic, then there are only finitely many preperiodic points $\beta \in \overline{K}$ which are S -integral with respect to α .

1. INTRODUCTION

Let K be a number field with algebraic closure \overline{K} , let S be a finite set of places of K containing the archimedean places, and let $\alpha, \beta \in \overline{K}$. We say that β is S -integral relative to α if no conjugate of β meets any conjugate of α at primes lying outside of S . More precisely, this means that for any prime $v \notin S$ and any K -embeddings $\sigma : K(\alpha) \rightarrow \overline{K}_v$ and $\tau : K(\alpha) \rightarrow \overline{K}_v$, we have

$$\begin{cases} |\sigma(\beta) - \tau(\alpha)|_v \geq 1 & \text{if } |\tau(\alpha)|_v \leq 1; \text{ and} \\ |\sigma(\beta)|_v \leq 1 & \text{if } |\tau(\alpha)|_v > 1. \end{cases}$$

Note that this definition extends naturally to the case where α is the point at infinity. We say that β is S -integral relative to the point at infinity if $|\sigma(\beta)|_v \leq 1$ for all $v \notin S$ and all K -embeddings $\sigma : K(\beta) \rightarrow \overline{K}_v$. Thus, our S -integral points coincide with the usual S -integers when α is the point at infinity.

In [BIR], the following conjecture is made.

Conjecture 1.0.1 (Ih). *Let K be a number field, and let S be a finite set of places of K that contains all the archimedean places. If $\varphi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ is a nonconstant rational function of degree $d > 1$ and $\alpha \in \mathbb{P}^1(K)$ is non-preperiodic for φ , then there are at most finitely many preperiodic points $\beta \in \mathbb{P}^1(\overline{K})$ that are S -integral with respect to α .*

In [BIR], it is proved that this conjecture holds when φ is a multiplication-by- n (for $n \geq 2$) map on an \mathbb{G}_m or on an elliptic curve. Recently, Petsche [Pet07] has proved the conjecture in the case where the point α is in the v -adic Fatou set at every place of K . A similar problem, dealing with points in inverse images of a single point rather than with preperiodic points, has been treated by Sookdeo [Soo07].

In this paper, we show that this conjecture is true for Chebyshev polynomials. That is, we prove the following, where we note that α may lie on the Julia set.

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Theorem 1.0.2. *Let φ be a Chebyshev polynomial. Let K be a number field, and let S be a finite set of places of K , containing all the archimedean places. Suppose that $\alpha \in K$ is not of type $\zeta + \zeta^{-1}$ for any root of unity ζ . Then the following set*

$$\mathbb{A}_{\varphi, \alpha, S}^1 := \{x \in \overline{\mathbb{Q}} : x \text{ is } S\text{-integral with respect to } \alpha \text{ and is } \varphi\text{-preperiodic}\}$$

is finite.

This will follow easily from the following theorem.

Theorem 1.0.3. *Let $(x_n)_{n=1}^\infty$ be a nonrepeating sequence of preperiodic points for a Chebyshev polynomial φ . Then for any non-preperiodic α in a number field K and any place v of K , we have*

$$(1.1) \quad \hat{h}_v(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v,$$

where $\sigma : K(x_n)/K \hookrightarrow \overline{K}_v$ means that σ is an embedding of $K(x_n)$ into \overline{K}_v , fixing K , here and in what follows.

Indeed, we will prove Theorem 1.0.3 slightly more generally for any $\alpha \in K$ if $v \not\mid \infty$, while for any $\alpha \neq -2, 0$, or 2 if $v \mid \infty$. (Note that the proof of Proposition 4.1.3 actually works for any $\alpha \in [-2, 2] - \{-2, 0, 2\}$.) The proof of Theorem 1.0.2 is then similar to the proof for \mathbb{G}_m given in [BIR]. Specifically, the proof of Theorem 1.0.3 breaks down into various cases, depending on whether or not the place v is finite or infinite and whether or not the point α is in the Julia set at v . The fact that the invariant measure for Chebyshev polynomials is not uniform on $[-2, 2]$ provides a slight twist.

The proof of Theorem 1.0.3 is fairly simple when v is nonarchimedean. Likewise, when v is archimedean but α is not in the Julia set at v , the proof follows almost immediately from an equidistribution result for continuous functions (see [Bil97]). When v is archimedean and α is in the Julia set at v , however, the proof becomes quite a bit more difficult. In particular, it is necessary to use A. Baker's theorem on linear forms in logarithms (see [Bak75]). We note that in all cases, our techniques are similar to those of [BIR].

The derivation of Theorem 1.0.2 from Theorem 1.0.3 goes as follows: suppose, for contrary, that Theorem 1.0.2 were to be false. Then we may further assume that the sequence $(x_n)_{n=1}^\infty$ in Theorem 1.0.3 is a sequence of α -integral points. Then we have

$$\begin{aligned} 0 &< \hat{h}(\alpha) \\ &= \sum_{\text{places } v \text{ of } K} \hat{h}_v(\alpha) \\ &= \sum_{\text{places } v \text{ of } K} \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v \\ &= \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\text{places } v \text{ of } K} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v \\ &= 0, \end{aligned}$$

where the equality on the third line comes from Theorem 1.0.3, the integrality hypothesis on the x_n enables us to switch $\sum_{\text{places } v \text{ of } K}$ and $\lim_{n \rightarrow \infty}$ to get the

equality on the fourth line, and the last equality is immediate from the product formula. This is a contradiction.

2. PRELIMINARIES

2.1. The Chebyshev polynomials.

Definition.

$$P_1(z) := z, \quad P_2(z) := z^2 - 2; \quad \text{and}$$

$$P_{m+1}(z) + P_{m-1}(z) = zP_m(z) \quad \text{for all } m \geq 2.$$

Then a *Chebyshev polynomial* is defined to be any of the P_m ($m \geq 2$).

These polynomials satisfy the following properties (see [Mil99, Section 7]).

- (1) For any $m \geq 1$, $P_m(\omega + \omega^{-1}) = \omega^m + \omega^{-m}$, equivalently $P_m(2 \cos \theta) = 2 \cos(m\theta)$, where $\omega \in \mathbb{C}^\times$ and $\theta \in \mathbb{R}$.
- (2) For any $\ell, m \geq 1$, $P_\ell \circ P_m = P_{\ell m}$.
- (3) For any $m \geq 3$, P_m has $m - 1$ distinct critical points in the finite plane, but only two critical values, i.e., ± 2 .

2.2. The dynamical systems of Chebyshev polynomials.

Definition. Let φ be a Chebyshev polynomial. The dynamical system induced by φ on \mathbb{P}^1 (or \mathbb{A}^1) is called the (*Chebyshev*) *dynamical system* with respect to φ or the φ -*dynamical system*. If φ is clearly understood from the context, we simply call it a *Chebyshev dynamical system* without reference to φ .

Proposition 2.2.1. *For any Chebyshev polynomial φ , the Julia set of the dynamical system induced by φ (resp. $-\varphi$) is $[-2, 2]$, which is naturally identified as a subset of the real line on the complex plane.*

Proof. See [Mil99, Section 7].

Proposition 2.2.2. *Let φ be a Chebyshev polynomial. Then the finite preperiodic points of the φ -dynamical system are the elements of \overline{K} of the form $\zeta + \zeta^{-1}$, where ζ is a root of unity.*

Proof. Take an element $z \in \overline{K}$. Then there is some $a \in \overline{K}$ such that $z = a + \frac{1}{a}$, as can be seen by finding a such that $a^2 - az + 1 = 0$. Note that a cannot be zero. Now if a is not a root of unity, then there is some place w of $K(a)$ such that $|a|_w > 1$. Thus, letting $m = \deg \varphi$ (≥ 2), we have

$$|\varphi^k(z)|_w = \left| a^{m^k} + \frac{1}{a^{m^k}} \right|_w > |a|_w^{m^k} - 1,$$

so $|\varphi^k(z)|_w$ goes to infinity as $k \rightarrow \infty$. Hence z cannot be preperiodic.

Conversely, if $z = \zeta + \zeta^{-1}$, where ζ is a root of unity then there are some positive integers $j \neq k$ such that $\zeta^{m^k} = \zeta^{m^j}$, which gives

$$\varphi^k(z) = \zeta^{m^k} + \frac{1}{\zeta^{m^k}} = \varphi^j(z),$$

so z is preperiodic for φ . □

2.3. The canonical height attached to a dynamical system. Let φ be a Chebyshev polynomial of degree m and let v be a place of a number field K . We define the local canonical height $\hat{h}_v(\alpha)$ of a point $\alpha \in \overline{K}_v$ associated to φ at any place v of K as

$$(2.2) \quad \hat{h}_v(\alpha) = \lim_{k \rightarrow \infty} \frac{\log \max(|\varphi^k(\alpha)|_v, 1)}{m^k}.$$

This local canonical height has the property that

$$\hat{h}_v(\varphi(\alpha)) = m\hat{h}_v(\alpha)$$

for any $\alpha \in \overline{K}_v$ (see [CG97] for details). Note that if v is a nonarchimedean place, then the Chebyshev dynamical system has good reduction at v and we have $\hat{h}_v(\alpha) = \log \max(|\alpha|_v, 1)$.

When $\alpha \in \overline{K}$, we have

$$(2.3) \quad \hat{h}(\alpha) = \sum_{\text{places } v \text{ of } K} \hat{h}_v(\alpha),$$

where the left-hand side is the (global) canonical height of α associated to φ .

In the case of the places $v \mid \infty$, we will use the φ -invariant measure $\mu_v := \mu_{v,\varphi}$ (see [Lyu83]) for φ to calculate these local heights. It is worth noticing this is not a uniform measure on $[-2, 2]$, unlike in the case of the dynamical system on \mathbb{P}^1 with respect to the map $z \mapsto z^2$, in which case the measure at archimedean places is the uniform probability Haar measure on the unit circle centered at the origin (see [Bil97]). The measure has more mass toward the end/boundary points ± 2 of the Julia set $[-2, 2]$. Further, the kernel $\frac{1}{\pi} \frac{1}{\sqrt{4-x^2}}$ has singularities at the extreme points ± 2 .

When $v \mid \infty$, we have the following formula for the local height at v (see [PST05, Appendix B] or [FRL06]) for any $\alpha \in \mathbb{C}$:

$$(2.4) \quad \hat{h}_v(\alpha) = \int_{\mathbb{C}} \log |z - \alpha|_v d\mu(z),$$

where $\mu := \mu_{v,\varphi}$ is the unique φ -invariant measure with support on the Julia set of φ at v .

Since any root of unity ξ_k , say $e^{2\pi i/k}$, is preperiodic for the the map sending z to z^m , we see that $\xi_k + \xi_k^{-1} = 2 \cos(2\pi/k)$ is preperiodic for φ . Now, the preperiodic points of φ are equidistributed with respect to μ (see [Lyu83, BH05]), so for any continuous function f on $[-2, 2]$ we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k f(2 \cos(j\pi/k)) = \int_{\mathbb{C}} f d\mu.$$

Thus $d\mu$ is the push-forward of the the uniform distribution on $[0, \pi]$ under the map $\theta \mapsto 2 \cos \theta$, thus

$$d\mu(x) = \frac{1}{\pi} \frac{d}{dx} \cos^{-1}(x/2) dx = \frac{1}{\pi} \frac{1}{\sqrt{4-x^2}} dx.$$

Thus, (2.4) becomes

$$(2.5) \quad \hat{h}_v(\alpha) = \frac{1}{\pi} \int_{-2}^2 \frac{1}{\sqrt{4-x^2}} \log |x - \alpha|_v dx$$

for any $\alpha \in \mathbb{C}$.

3. ARCHIMEDEAN PLACES

3.1. A counting lemma. Let K be a number field, and let $I \subset [-2, 2]$ be an interval. For any root of unity $\zeta \in \overline{K}$, write $x_\zeta := \zeta + \zeta^{-1}$. Let

$$\mathcal{N}(x_\zeta, I) := \#\{\sigma(x_\zeta) \in I : \sigma \in \text{Gal}(K(x_\zeta)/K)\}.$$

Lemma 3.1.1. *Keep notation just above. Let $-2 \leq c < d \leq 2$, and let $I := (c, d]$ be an interval. Then for any real $0 < \gamma < 1$ and any root of unity $\zeta \in \overline{K}$,*

$$(3.6) \quad \mathcal{N}(x_\zeta, I) = \frac{[K(x_\zeta) : K]}{\pi} \left(\cos^{-1} \frac{c}{2} - \cos^{-1} \frac{d}{2} \right) + O_\gamma([K(x_\zeta) : K]^\gamma)$$

where $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$ is the arccos function. In particular, when $-2 < c < d < 2$, we may write

$$(3.7) \quad \mathcal{N}(x_\zeta, I) \leq M[K(x_\zeta) : K](d - c) + O_\gamma([K(x_\zeta) : K]^\gamma)$$

where $M := M_{c,d}$ is the supremum of $\frac{1}{\sqrt{4-x^2}}$ on $(c, d]$.

Proof. Write $\zeta = e^{2\pi i \frac{a}{N}}$, where N is a positive integer and $1 \leq a \leq N$. Then note

$$\begin{aligned} x_\zeta \in I &\iff e^{2\pi i \frac{a}{N}} + e^{-2\pi i \frac{a}{N}} \in I \\ &\iff \cos\left(2\pi \frac{a}{N}\right) \in \left(\frac{c}{2}, \frac{d}{2}\right] \\ &\iff a \in \frac{N}{2\pi} \left[\cos^{-1} \frac{d}{2}, \cos^{-1} \frac{c}{2}\right]. \end{aligned}$$

Then (3.6) follows immediately from [BIR, Prop. 1.3]. To see that (3.7) holds, note that the derivative of the function $\cos^{-1}(x/2)$ is $\frac{1}{\sqrt{4-x^2}}$. Thus, (3.7) is a consequence of (3.6) and along with the Mean Value Theorem from calculus. \square

Remark. In the above, more precisely, we may define M to be the supremum of $\frac{1}{\pi \sqrt{4-x^2}}$ on $(c, d]$. However, this difference will not matter for our later purpose. So we will keep the above choice for M .

3.2. Baker's lower bounds for linear forms in logarithms. Here we state the theorem on Baker's lower bounds for linear forms in logarithms, (see [Bak75][A. Baker, Thm. 3.1, p. 22]).

Theorem 3.2.1 (Baker). *Suppose that $e^{2\pi i \theta_0} \in \overline{\mathbb{Q}}$. Then there exists a constant $C := C(\theta_0) > 0$ such that for any coprime $a, N \in \mathbb{Z}$ ($N \neq 0$ or ± 1) with $\frac{a}{N} \neq \theta_0$,*

$$\left| \frac{a}{N} - \theta_0 \right| \geq M^{-C}$$

where $M := \max(|a|, |N|)$.

Proof.

$$\left| \frac{a}{N} - \theta_0 \right| = \frac{1}{2\pi} \left| \frac{a}{N} \cdot 2\pi i - 2\pi i \theta_0 \right|.$$

Then apply Baker's theorem to the absolute value of the right hand side and adjust the resulting constant for $\frac{1}{2\pi}$. (Also recall that $N \neq 0$ or ± 1 .) \square

4. THE MAIN THEOREM AND ITS VARIANT

4.1. The main theorem and its proof. We will prove Theorem 1.0.3 by breaking it into several cases. We begin with the case where the place v is finite. For the sake of precision, we will state when we need α to be in K and when it suffices that it be in \overline{K}_v .

Proposition 4.1.1. *Let $(\zeta_n)_{n=1}^\infty$ be a sequence of distinct roots of unity, and write $x_n := \zeta_n + \zeta_n^{-1}$ for any $n \geq 1$. If v is finite, then for any $\alpha \in \overline{K}_v$, we have*

$$(4.8) \quad \hat{h}_v(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v.$$

Proof. If $|\alpha|_v > 1$, then $|\sigma(x_n) - \alpha|_v = |\alpha|_v$. Thus, (4.8) is immediate. Now, suppose that $|\alpha|_v \leq 1$. Let $r < 1$ be a real number. Let x_m and x_n satisfy that $|x_m - \alpha|_v \leq r$ and $|x_n - \alpha|_v \leq r$. Then observe

$$\begin{aligned} r &\geq |(x_m - \alpha) - (x_n - \alpha)|_v \\ &= |x_m - x_n|_v \\ &= \left| (\zeta_m - \zeta_n) - \frac{\zeta_m - \zeta_n}{\zeta_m \zeta_n} \right|_v \\ &= |\zeta_m - \zeta_n|_v |1 - (\zeta_m \zeta_n)^{-1}|_v \\ &= |1 - \zeta_m^{-1} \zeta_n|_v |1 - (\zeta_m \zeta_n)^{-1}|_v. \end{aligned}$$

Hence either $|1 - \zeta_m^{-1} \zeta_n|_v \leq \sqrt{r}$ or $|1 - (\zeta_m \zeta_n)^{-1}|_v \leq \sqrt{r}$. In the first (resp. second) case it follows that $\zeta_m^{-1} \zeta_n$ (resp. $(\zeta_m \zeta_n)^{-1}$) must have order equal to a power of the prime number $\in \mathbb{Z}$ lying below v , and that there are only finitely many choices for $\zeta_m^{-1} \zeta_n$ (resp. $(\zeta_m \zeta_n)^{-1}$) in the first (resp. second) case. Thus, for any real $r < 1$, there are only finitely many indices $n \geq 1$ such that $|x_n - \alpha|_v \leq r$, which immediately implies the desired convergence in this case. \square

We now treat the archimedean v for which α is outside the Julia set at v .

Proposition 4.1.2. *Let x_n be as in Proposition 4.1.1. If v is archimedean, then for any $\alpha \in \mathbb{C} - [-2, 2]$, we have*

$$\hat{h}_v(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v.$$

Proof. From (2.4), we have

$$\int_{\mathbb{C}} \log |z - \alpha|_v d\mu_v(z) = \hat{h}_v(\alpha),$$

where $\mu_v := \mu_{v, \varphi}$ is the invariant measure for φ at v . This measure is supported on $[-2, 2]$, so if g is a function on \mathbb{C} that agrees with $\log |z - \alpha|_v$ on $[-2, 2]$ we have

$$(4.9) \quad \int_{\mathbb{C}} g(z) d\mu_v(z) = \hat{h}_v(\alpha).$$

Let $\epsilon = \min_{w \in [-2, 2]} |w - \alpha|_v$ (note that $\epsilon \neq 0$ since $\alpha \notin [-2, 2]$) and define $g(z)$ as

$$g(z) = \min \left(\log \max(|z - \alpha|_v, \epsilon), \log(|\alpha|_v + 2) \right).$$

Then g is continuous and bounded on all of \mathbb{C} and agrees with $\log|z - \alpha|$ on $[-2, 2]$. By [Bil97, Theorem 1.1], we have

$$\int_{\mathbb{C}} g(z) d\mu_v(z) = \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} g(\sigma(x_n)).$$

Since all $x_n \in [-2, 2]$, this finishes the proof, using (4.9). \square

Now, we come to the most difficult case.

Proposition 4.1.3. *Let x_n be as in Proposition 4.1.1. If $v|\infty$ and $\alpha \in [-2, 2]$ is not preperiodic, then we have*

$$\hat{h}_v(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log|\sigma(x_n) - \alpha|_v.$$

Proof. We may assume that $\alpha \in K$. If v is archimedean and $\alpha \in K$ is in $[-2, 2]$, then we have $\hat{h}_v(\alpha) = 0$. This follows from the fact that φ maps $[-2, 2]$ to itself, so if $\alpha \in [-2, 2]$, then $|\varphi^n(\alpha)|_v$ is bounded for all n , so $\hat{h}_v(\alpha) = 0$ by (2.2).

Note $|x|_v = |\tau(x)|$ for all $x \in \mathbb{C}$, where $\tau : K(x)/K \hookrightarrow \mathbb{C}$ is associated to v and $|\cdot|$ is the usual absolute value on \mathbb{C} . To simplify our notation, we will fix one $v|\infty$, suppress v in the notation of the absolute value, and use $|\cdot|$ according to this observation, i.e., without loss of generality we will prove this theorem for the place v equal to the usual absolute value ($|z| = \sqrt{z\bar{z}}$, $z \in \mathbb{C}$). However, we will keep v in the notation of the local height \hat{h}_v to avoid any confusion with the global height \hat{h} .

We may write $\alpha = e^{2\pi i\theta_0} + e^{-2\pi i\theta_0} = 2\cos(2\pi\theta_0)$, where $\theta_0 \in (-\frac{1}{2}, \frac{1}{2}]$. Note α cannot be equal to $-2, 2$, or 0 since we assume that α is not preperiodic. Note that $\int_0^\epsilon \log\left(\frac{t}{\epsilon}\right) dt = -\epsilon$ for any $\epsilon > 0$.

Write

$$\begin{aligned} x &= e^{2\pi i\theta} + e^{-2\pi i\theta} = 2\cos(2\pi\theta); & \text{and} \\ x_n &= e^{2\pi i\frac{a}{N}} + e^{-2\pi i\frac{a}{N}} = 2\cos\left(2\pi\frac{a}{N}\right) \end{aligned}$$

where a and $N(\neq 0)$ are integers (depending on $n \geq 1$), and $|\frac{a}{N}| \leq 1$.

We recall that $\hat{h}_v(\alpha) = 0$ since for any α in $[-2, 2]$, the quantity $|P_m^k(\alpha)|$ is bounded for all $k \geq 1$. Thus, we have

$$(4.10) \quad \frac{1}{\pi} \int_{-2}^2 \frac{1}{\sqrt{4-x^2}} \log|x - \alpha| dx = 0.$$

Hence it will suffice to show, for all $n \gg 1$, that the quantity

$$(4.11) \quad \frac{1}{[K(x_n) : K]} \left| \sum_{\sigma: K(x_n)/K \hookrightarrow \mathbb{C}} \log|\sigma(x_n) - \alpha| \right|$$

can be made sufficiently small.

Fix $\epsilon > 0$. By (4.10), we have

$$(4.12) \quad \left| \int_{\alpha-\delta}^{\alpha+\delta} \frac{1}{\sqrt{4-x^2}} \log|x - \alpha| dx \right| < \epsilon,$$

i.e., sufficiently small for all sufficiently small $\delta > 0$.

Let $g_\delta(z) = \log \max(|z - \alpha|, \delta)$. By (4.12) and the fact that $0 > g_\delta(x) > \log |x - \alpha|$ for $x \in [\alpha - \delta, \alpha + \delta]$, we see that

$$(4.13) \quad \left| \int_{-2}^2 \frac{1}{\sqrt{4-x^2}} g_\delta(x) dx \right| < \epsilon.$$

By the equidistribution theorem of Baker/Rumely ([BR06]), Chambert-Loir ([CL06]) and Favre/ Rivera-Letelier ([FRL06]), we see that for all sufficiently large n , the quantity

$$(4.14) \quad \left| \left(\frac{1}{\pi} \int_{-2}^2 \frac{1}{\sqrt{4-x^2}} g_\delta(x) dx \right) - \left(\frac{1}{[K(x_n) : K]} \sum_{\sigma: K(x_n)/K \hookrightarrow \mathbb{C}} g_\delta(\sigma(x_n)) \right) \right|$$

is sufficiently small. Thus, by (4.13) it suffices to show that

$$\frac{1}{[K(x_n) : K]} \left| \sum_{\sigma: K(x_n)/K \hookrightarrow \mathbb{C}} (\log |\sigma(x_n) - \alpha| - g_\delta(\sigma(x_n))) \right|$$

is sufficiently small for all $n \gg 1$ and all sufficiently small $\delta > 0$. Since $\log |\sigma(x_n) - \alpha| = g_\delta(\sigma(x_n))$ outside of $[\alpha - \delta, \alpha + \delta]$, it in turn it suffices to show that

$$(4.15) \quad \frac{1}{[K(x_n) : K]} \left| \sum_{\substack{\sigma: K(x_n)/K \hookrightarrow \mathbb{C} \\ \sigma(x_n) \in [\alpha - \delta, \alpha + \delta]}} (\log |\sigma(x_n) - \alpha| - g_\delta(\sigma(x_n))) \right|$$

is sufficiently small for all $n \gg 1$ and all sufficiently small $\delta > 0$.

Now, when $\delta > 0$ is small and x is in $[\alpha - \delta, \alpha + \delta]$, we have $0 > g_\delta(x) \geq \log |x - \alpha|$ and the quantity (4.15) is bounded above by

$$(4.16) \quad \frac{1}{[K(x_n) : K]} \left| \sum_{\substack{\sigma: K(x_n)/K \hookrightarrow \mathbb{C} \\ \sigma(x_n) \in [\alpha - \delta, \alpha + \delta]}} \log |\sigma(x_n) - \alpha| \right|.$$

Hence, finally it suffices to show that (4.16) is sufficiently small whenever n is sufficiently large and $\delta > 0$ is sufficiently small.

If we choose $\delta > 0$ sufficiently small, we may assume that

$$(4.17) \quad \min_{x \in [\alpha - \delta, \alpha + \delta]} \left(\frac{1}{\sqrt{4-x^2}} \right) \geq \frac{1}{2} \max_{x \in [\alpha - \delta, \alpha + \delta]} \left(\frac{1}{\sqrt{4-x^2}} \right).$$

We define M as

$$(4.18) \quad M := \max_{x \in [\alpha - \delta, \alpha + \delta]} \left(\frac{1}{\sqrt{4-x^2}} \right)$$

Choose a large positive integer D . For any $1 \leq i \leq D$ denote by S_i the interval

$$[\alpha - \delta + (i-1)(\delta/D), \alpha - \delta + i(\delta/D)].$$

Given any $n \gg 1$, let $N_i := N_i(n)$ denote the number of $\sigma(x_n)$'s belonging to S_i .

Note that $\log |\sigma(x_n) - \alpha| \leq 0$, whenever $\sigma(x_n)$ belongs to any of the S_i ($1 \leq i \leq D$). For any $1 \leq i \leq D-1$, on S_i we have

$$\begin{aligned} & \frac{1}{[K(x_n) : K]} \left| \sum_{\substack{\sigma: K(x_n)/K \hookrightarrow \mathbb{C} \\ \sigma(x_n) \in S_i}} \log |\sigma(x_n) - \alpha| \right| \\ & \leq M(\delta/D) \left| \log |(D-i)(\delta/D)| \right| + O \left(\frac{1}{\sqrt{[K(x_n) : K]}} \right) \left| \log ((D-i)\delta/D) \right| \\ & \quad (\text{by Lemma 3.1.1 with } \gamma = 1/2) \\ & \leq 2 \left| \int_{S_{i+1}} (M/2) \log |x - \alpha| dx \right| + O \left(\frac{1}{\sqrt{[K(x_n) : K]}} \right) \left| \log ((D-i)\delta/D) \right| \\ & \leq 2 \left| \int_{S_{i+1}} \frac{1}{\sqrt{4-x^2}} \log |x - \alpha| dx \right| + O \left(\frac{1}{\sqrt{[K(x_n) : K]}} \right) \left| \log ((D-i)\delta/D) \right| \\ & \quad (\text{by (4.17) and (4.18)}). \end{aligned}$$

Summing up over all $1 \leq i \leq D-1$ and applying (4.12) we obtain

$$(4.19) \quad \frac{1}{[K(x_n) : K]} \left| \sum_{\substack{\sigma: K(x_n)/K \hookrightarrow \mathbb{C} \\ \sigma(x_n) \in [\alpha - \delta, \alpha - (\delta/D)]}} \log |\sigma(x_n) - \alpha| \right| \leq 2\epsilon + \frac{1}{\sqrt{[K(x_n) : K]}} C_2 D (|\log(\delta/D)| + \log D),$$

for some constant $C_2 > 0$ independent of n and D .

Similarly, we see that

$$(4.20) \quad \frac{1}{[K(x_n) : K]} \left| \sum_{\substack{\sigma: K(x_n)/K \hookrightarrow \mathbb{C} \\ \sigma(x_n) \in [\alpha + (\delta/D), \alpha + \delta]}} \log |\sigma(x_n) - \alpha| \right| \leq 2\epsilon + \frac{1}{\sqrt{[K(x_n) : K]}} C_3 D (|\log(\delta/D)| + \log D),$$

for some constant $C_3 > 0$ independent of n and D . Since $|\log(1/D)|$ and $\log D$ grow more slowly than any power of D , we see that quantities (4.19) and (4.20) can be made sufficiently small when D is large and $[K(x_n) : K] \geq D^4$.

Now, for all sufficiently small $\delta > 0$, we have

$$0 \geq \log |x_n - \alpha| = \log \left| 2 \cos \left(2\pi \frac{a}{N} \right) - 2 \cos(2\pi\theta_0) \right| \geq \log \left| \frac{a}{N} - \theta_0 \right| + O(1)$$

for all $x_n \in [\alpha - \delta, \alpha + \delta]$. When N is sufficiently large, Theorem 3.2.1 thus yields

$$0 \geq \log |x_n - \alpha| \geq -C_4 \log N + O(1)$$

where $C_4 > 0$ is a constant independent of n . This inequality is true not only for x_n itself, but also for all its K -Galois conjugates that belong to $[\alpha - \delta, \alpha + \delta]$, i.e., after readjusting C_4 if necessary, we have

$$(4.21) \quad 0 \geq \log |\sigma(x_n) - \alpha| \geq -C_4 \log N$$

for all $\sigma(x_n) \in [\alpha - \delta, \alpha + \delta]$, where $C_4 > 0$ is a constant independent of (all) $n \gg 1$. Thus, it follows from (3.7) (again with $\gamma = 1/4$) that we have

$$(4.22) \quad \frac{1}{[K(x_n) : K]} \left| \sum_{\substack{\sigma: K(x_n)/K \hookrightarrow \mathbb{C} \\ \sigma(x_n) \in [\alpha - (\delta/D), \alpha + (\delta/D)]}} \log |\sigma(x_n) - \alpha| \right| \\ \leq C_4 M(\delta/D) \log N + \frac{C_5 \log N}{[K(x_n) : K]^{1/2}}$$

where $C_5 > 0$ is a constant.

Write ϕ for the Euler function, and suppose that $N \gg 1$. Note that

$$[K(x_n) : K] \geq \frac{[\mathbb{Q}(x_n) : \mathbb{Q}]}{[K : \mathbb{Q}]} = \frac{\phi(N)}{[K : \mathbb{Q}]}$$

and $\phi(N) \geq \sqrt{N}$ (see [HW79, page 267, Thm 327]), and hence that $[K(x_n) : K]^{1/2} \gg \sqrt[4]{N}$. Now, let $D = \lfloor \sqrt[4]{N} \rfloor$. (Note this choice of D is compatible with that of D in (4.19) and (4.20).) Then, when N is sufficiently large, the right-hand sides of (4.19) and (4.20) are both sufficiently small and the right-hand side of (4.22) is also sufficiently small. Combining equations (4.19), (4.20), and (4.22) we then obtain that

$$\frac{1}{[K(x_n) : K]} \left| \sum_{\sigma: K(x_n)/K \hookrightarrow \mathbb{C}} \log |\sigma(x_n) - \alpha| \right| \text{ is sufficiently small.}$$

Thus, we must have $\lim_{n \rightarrow \infty} \sum_{\sigma: K(x_n)/K \hookrightarrow \mathbb{C}} \log |\sigma(x_n) - \alpha| = 0$, as desired. \square

The proof of Theorem 1.0.3 is now immediate since the Propositions above cover all v and all non-preperiodic $\alpha \in K$. Now, we are ready to prove our main theorem, Theorem 1.0.2.

Proof of Theorem 1.0.2. Let S be a finite set of places of K that includes all the archimedean places. After extending S to a larger finite set if necessary, which only makes the set $\mathbb{A}_{\varphi, \alpha, S}^1$ larger, we may assume that S also contains all the places v for which $|\alpha|_v > 1$. Then for any $v \notin S$ and any preperiodic point x_n we have

$$(4.23) \quad \log |\sigma(x_n) - \alpha|_v = 0 \text{ for any embedding } \sigma : K(x_n)/K \longrightarrow \overline{K}_v.$$

Assume that $(x_n)_{n=1}^\infty$ is an infinite nonrepeating sequence in $\mathbb{A}_{\varphi, \alpha, S}^1$. Since we can interchange a limit with a finite sum, we have

$$(4.24) \quad \frac{1}{[K : \mathbb{Q}]} \hat{h}(\alpha) = \sum_{v \in S} \lim_{n \rightarrow \infty} \frac{1}{[K(x_n) : \mathbb{Q}]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v$$

(by (2.3), (4.23), and Thm. 1.0.3)

$$= \lim_{n \rightarrow \infty} \sum_{v \in S} \frac{1}{[K(x_n) : \mathbb{Q}]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v \text{ (switching sum and limit)}$$

$$= \lim_{n \rightarrow \infty} \sum_{\text{places } v \text{ of } K} \frac{1}{[K(x_n) : \mathbb{Q}]} \sum_{\sigma: K(x_n)/K \hookrightarrow \overline{K}_v} \log |\sigma(x_n) - \alpha|_v \text{ (by (4.23))}$$

$$= 0 \quad \text{(by the product formula).}$$

Since α is not preperiodic, however, we have $\hat{h}(\alpha) > 0$. Thus, we have a contradiction, so $\mathbb{A}^1_{\varphi, \alpha, S}$ must be finite. \square

4.2. A variant of the Chebyshev dynamical systems. We look at different Chebyshev polynomials defined by the following recursion formula:

$$Q_1(z) := z, \quad Q_2(z) := z^2 + 2; \quad \text{and} \\ Q_{m+1}(z) - Q_{m-1}(z) = zQ_m(z) \quad \text{for all } m \geq 2.$$

The dynamical system induced by any of the Q_m ($m \geq 2$) on \mathbb{A}^1 (or \mathbb{P}^1) has properties similar to those for the Chebyshev dynamical systems, for instance:

- (i) The Julia set is equal to the interval $[-2, 2]$ on the y -axis;
- (ii) The preperiodic points are (either ∞ or) the points of type $\zeta - \zeta^{-1}$, where ζ is a root of unity.
- (iii) The corresponding measures μ_v satisfy

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |z - \alpha|_v d\mu_v = \begin{cases} \log \max\{|\alpha|_v, 1\}, & \text{if } v \neq \infty; \\ \frac{1}{\pi} \int_{-2}^2 \frac{1}{\sqrt{4-y^2}} \log |yi - \alpha|_v dy, & \text{otherwise} \end{cases}$$

where $\alpha \in K$ (K a number field), v is a place of K , and dy is the usual Lebesgue measure on $[-2, 2]$. Note that the measure μ_v ($v|\infty$) is supported on the interval $[-2, 2]$ on the y -axis.

It is then easy to see that arguments similar to the above prove the following:

Theorem 4.2.1. *Let ψ be any of the Q_m ($m \geq 2$). Let K be a number field, and let S be a finite set of places of K , containing all the infinite ones. Suppose that $\alpha \in K$ is not of type $\zeta - \zeta^{-1}$ for any root of unity ζ . Then the following set*

$$\mathbb{A}^1_{\psi, \alpha, S} := \{z \in \overline{\mathbb{Q}} : z \text{ is } S\text{-integral with respect to } \alpha \text{ and is } \psi\text{-preperiodic}\}$$

is finite.

REFERENCES

- [Bak75] A. Baker, *Transcendental number theory*, Cambridge University Press, Cambridge, 1975.
- [BH05] M. Baker and L.-C. Hsia, *Canonical heights, transfinite diameters, and polynomial dynamics*, J. Reine Angew. Math. (2005), to appear. Available at [arxiv:math.NT/0305181](https://arxiv.org/abs/math/0305181), 28 pages.
- [Bil97] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math **89** (1997), 465–476.
- [BIR] M. Baker, S. Ih, and R. Rumely, *A finiteness property of torsion points*, to appear in *Algebra and Number Theory*, available at [arxiv:math.NT/0509485](https://arxiv.org/abs/math/0509485), 30 pages.
- [BR06] M. Baker and R. Rumely, *Equidistribution of small points, rational dynamics, and potential theory*, Ann. Inst. Fourier **56** (2006), no. 3, 625–688.
- [CG97] G. S. Call and S. W. Goldstone, *Canonical heights on projective space*, J. Number Theory **63** (1997), no. 2, 211–243.
- [CL06] A. Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235.
- [FRL06] C. Favre and J. Rivera-Letelier, *Équidistribution quantitative des points de petite hauteur sur la droite projective*, Math. Ann. **335** (2006), no. 2, 311–361.
- [HW79] Godfrey H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 5th ed., Oxford University Press, Oxford, 1979.
- [Lyu83] M. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems **3** (1983), 351–385.
- [Mil99] J. Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, Introductory lectures.
- [Pet07] C. Petsche, *S-integral preperiodic points in dynamical systems*, in preparation, 2007.

- [PST05] J. Piñeiro, L. Szpiro, and T. J. Tucker, *Mahler measure for dynamical systems on \mathbf{P}^1 and intersection theory on a singular arithmetic surface*, Geometric Methods in Algebra and Number Theory (Fedor Bogomolov and Yuri Tschinkel, eds.), Progress in Mathematics, vol. 235, Birkhäuser, 2005, pp. 219–250. (Available at <http://math.gc.cuny.edu/faculty/szpiro/504miami.pdf>).
- [Soo07] V. Sookdeo, *Integral points in inverse images*, in preparation, 2007.

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